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Contents

Chapter 1

FUNCTION AND LIMITS

EXERCISE 1.1

SHORT ANSWERS TO THE QUESTIONS

Q.1 Given that $f(x) = \sqrt{x+4}$, then find $f(x-1)$.

Sol: $x = \sqrt{A}$ put in (2) $P = 4\sqrt{A}$

Q.2 Express the area A of a circle as a function of its circumference C. *(Lahore Board 2017)*

(Bahawalpur , Rawalpindi Board 2019)

Sol: Let r be the radius of the circle.

So, its area $= A = \pi r^2$ ..... (1)

Its circumference $= C = 2\pi r$ ..... (2)

From (2) $r = \frac{C}{2\pi}$ put $r = \frac{C}{2\pi}$ in (1)

$$A = \pi\left(\frac{C}{2\pi}\right)^2 = \pi\left(\frac{C^2}{4\pi^2}\right) = \frac{C^2}{4\pi} \Rightarrow A = \frac{1}{4\pi}C^2$$

Q.3 Express the volume V of a cube as a function of the area A of its base.

(Sargodha Board 2017)(A.J.K Board 2017)

Sol: Let x be the side of the cube.

So, its volume $= V = x \cdot x \cdot x = x^3$ ..... (1)

Area of base $= A = x \cdot x = x^2$ ..... (2)

From (2) $x = \sqrt{A}$ put in (1) $\Rightarrow V = (A^{1/2})^3 = A^{3/2}$

Q.4 Show that $x = at^2$, $y = 2at$ are parametric equations of parabola $y^2 = 4ax$.

(Lahore Board 2017 G-I)

Sol: $x = at^2$ ... (i)

$y = 2at$ ... (ii)

$y^2 = 4ax$ ... (iii)

Putting equation (i) and equation (ii) in equation (iii)

$(2at)^2 = 4a(at^2)$

$4a^2t^2 = 4a^2t^2$

Which is true. Therefore equation (i) and equation (ii) are parametric equations of parabola.

$y^2 = 4ax$

Q.5 Determine that function is even or odd if

$f(x) = x^{\frac{2}{3}} + 6$

(Multan Board 2013 G-II)(Gujranwala Board 2011)

(Lahore Board 2015 G-I)(Sargodha Board 2017)

Sol: $f(-x) = (-x)^{\frac{2}{3}} + 6 = ((-x)^2)^{\frac{1}{3}} + 6 = (x^2)^{\frac{1}{3}} + 6$

$= x^{\frac{2}{3}} + 6 = f(x)$

Thus f(x) is even

Q.6 Determine whether $f(x) = \frac{x^3 - x}{x^2 + 1}$ is even or odd.

Sol: $f(x) = \frac{x^3 - x}{x^2 + 1}$

Replace x by $-x$

$$f(-x) = \frac{(-x)^3 - (-x)}{(-x)^2 + 1}$$

$$= -\frac{x^3 + x}{x^2 + 1} = -\frac{(x^3 - x)}{x^2 + 1} = -f(x)$$

$\therefore$ It is an odd function

LONG QUESTIONS

Q.1 Show that the parametric equation.

$x = a\cos\theta$, $y = b\sin\theta$ represent the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *(Multan Board 2014 G-II)*

Sol: The parametric equations are.

$x = a\cos\theta$..... (1)

$y = b\sin\theta$..... (2)

We eliminate the parameter "θ" from equation (1) and (2).

By squaring we get.

$$x^2 = a^2\cos^2\theta \Rightarrow \frac{x^2}{a^2} = \cos^2\theta \quad \text{..... (3)}$$

$$y^2 = b^2\sin^2\theta \Rightarrow \frac{y^2}{b^2} = \sin^2\theta \quad \text{..... (4)}$$

eq. (3) + eq. (4) we get.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Which is the equation of ellipse

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 Which of the following is an odd function? *(D.G.K Board 2014 G-II)*

(a) cos x (b) cos hx
(c) sin hx (d) $\sin^2 x$

Q.2 If y is image of x under function f, we write it as: *(D.G.K Board 2014 G-II)*

(a) x = f(y) (b) y ≠ f(x)
(c) y = f(x) (d) y = x

Q.3 If f(x) = 2x + 5, then f(2) equals: *(D.G.K Board 2014 G-I)*

(a) 1 (b) 9
(c) –9 (d) –1

Q.4 The function is said to be an even function if f(–x) = ———

(a) –f(x) (b) f(–x)
(c) f(x) (d) None of these

Q.5 If f(–x) = f(x), then f(x) is called: *(Multan Board 2014 G-I)*

(a) Odd function (b) Even function
(c) Explicit function (d) Implicit function

Q.6 f(x) = cos x + sin x is ——— function: *(Bahawalpur Board 2014)(Faisalabad Board 2016)*

(a) Even (b) Odd
(c) Both even & odd (d) Neither the even nor odd

Q.7 If f(x) = cos x, then f(0) = ? *(Rawalpindi Board 2014)*

(a) –1 (b) $-\frac{1}{2}$
(c) 0 (d) 1

Q.8 2 sin hx is equal:

(a) $e^x + e^{-x}$ (b) $e^x - e^{-x}$
(c) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$ (d) $\frac{e^x + e^{-x}}{e^x - e^{-x}}$

Q.9 A function defined by f(x) = x^2 is: *(Sahiwal Board 2013)*

(a) Even function (b) Odd function
(c) Identity function (d) Linear function

Q.10 $x = at^2$, y = 2at are parametric equations of curve *(Sahiwal Board 2013)*

(a) Circle (b) Parabola
(c) Hyperbola (d) Ellipse

Q.11 x = a cos θ, y = b sin θ are parametric equation of: *(Multan Board 2013 G-II)*

(a) Parabola (b) Ellipse
(c) Circle (d) Hyperbola

Q.12 If f(x) is a function such that f(–x) = f(x), then f(x) is said to be:

(a) Odd (b) Even
(c) Constant (d) Linear

Q.13 A function A:X → Y defined by A(X) = a, ∀ x ∈ X and a ∈ Y and fixed is called: *(Faisalabad 2013)*

(a) Linear function (b) Identity function
(c) Constant function (d) Non linear function

Q.14 If f(x) = x^3, then f(2) = ? *(D.G.K Board 2013 G-II)*

(a) 8 (b) 2
(c) –2 (d) –8

Q.15 Equals x = 3 cos t, y = 3 sin t represent equation of: *(D.G.K Board 2013 G-II)*

(a) Line (b) Circle
(c) Parabola (d) Hyperbola

Q.16 The perimeter P of a square as a function of its area A: *(D.G.K Board 2012 G-II) (Lahore Board 2011 G-I)(Faisalabad Board 2017) (Bahawalpur Board 2019)*

(a) $P = \sqrt{A}$ (b) $P = 2\sqrt{A}$
(c) $P = 3\sqrt{A}$ (d) $P = 4\sqrt{A}$

Q.17 The term function was introduced by:

(a) Euler (b) Leibniz
(c) Cauchy (d) Newton

Q.18 The volume V of a cube as a function of the area A of its base is:

(a) $\sqrt{A^2}$ (b) A^3
(c) $\sqrt[3]{A^3}$ (d) $\sqrt{A^3}$

Q.19 If f(–x) = f(x) for all x in the domain of f, then f is:

(a) Constant function (b) Identity function
(c) Even function (d) Odd function

Q.20 The area A of a circle as a function of its circumference C is:

(a) $\frac{C^2}{2\pi}$ (b) $\frac{C^2}{4\pi}$
(c) $\frac{C^2}{\pi}$ (d) $\frac{C}{4\pi}$

Q.21 The symbol y = f(x) i.e. y is equal to f of x, invented by Swiss mathematician ———

(a) Euler (b) Cauchy
(c) Leibniz (d) Newton

Q.22 If a function f is from a set X to a set Y, then set X is called the ——— of f:

(a) Domain (b) Range
(c) Co-domain (d) None of these

Q.23 If f : R → R defined by f(x) = 2 ∀ x ∈ R is a/an ——— function:

(a) Exponential (b) Inverse
(c) Constant (d) Implicit

Q.24 If $f(x) = x^3 - 2x^2 + 4x - 1$, then $f(0) =$ ———
(a) 0 (b) 1
(c) −1 (d) 2

Q.25 The function $y = e^{x \ln 2} = 2x$ is a/an ——— function of x.
(a) Constant (b) Explicit
(c) Exponential (d) Logarithmic

Q.26 $y = \log_a x$, where $a > 0$ and $a \neq 1$ is called a ——— function of x:
(a) Implicit (b) Logarithmic
(c) Exponential (d) None of these

Q.27 A function $P(x) = 6x^4 + 7x^3 + 5x + 1$ is called a polynomial function of degree ——— with leading coefficient ———
(a) 4, 6 (b) 2, 7
(c) 2, 3 (d) 2, 5

Q.28 If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then y is a ——— of x:
(a) Independent variable
(b) Not function
(c) Function (d) None of these

Q.29 If $y = f(x)$, then the variable x is called ——— variable of a function f:
(a) Dependent (b) Independent
(c) Image of y (d) None of these

Q.30 If $f(x) = x^3 - 2x^2 + 4x - 1$, then $f(1) =$ ———
(a) 1 (b) 3
(c) −1 (d) 2

Q.31 If $f(x) = x^2$, then the range of f is ———
(a) Set of all non-negative real numbers
(b) Set of integers
(c) Set of real numbers
(d) Set of rational numbers

Q.32 If the degree of a polynomial function is ———, then it is called a linear function:
(a) 0 (b) 1
(c) 2 (d) 3

Q.33 cot h x = ——— is called hyperbolic cotangent function:
(a) $\frac{e^x + e^{-x}}{e^x + e^{-x}}$ (b) $\frac{2}{e^x - e^{-x}}$
(c) $\frac{e^x + e^{-x}}{e^x - e^{-x}}$ (d) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$

Q.34 cos h x = $\frac{e^x + e^{-x}}{2}$ is called ——— function:
(a) Cosine (b) Hyperbolic cosine
(c) Inverse cosine (d) None of these

Q.35 tan h x = ———
(a) $\frac{2}{e^x + e^{-x}}$ (b) $\frac{2}{e^x - e^{-x}}$
(c) $\frac{e^x + e^{-x}}{e^x - e^{-x}}$ (d) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$

Q.36 $y = \sqrt{x - 1}$ is a/an ——— function of x:
(a) Constant (b) Implicit
(c) Explicit (d) Inverse

Q.34 If $f(-x) = f(x)$, for every number x, $-x$ in the domain of f, then the function f is called a/an ——— function: *(D.G Khan Board 2017 G-I)*
(a) Constant (b) Linear
(c) Odd (d) Even

Q.38 $f(x) = x^3$ is a/an ——— function:
(a) Constant (b) Odd
(c) Even (d) None of these

Q.39 $\cos h^2 x + \sin h^2 x =$ ———
(a) sin h 2x (b) cos h 2x
(c) −1 (d) tan h x

Q.40 $f(x) = \cos x$ is a/an ——— function:
(a) Constant (b) Odd
(c) Even (d) None of these

Q.41 If y is easily expressed in terms of the independent variable x, then y is called a/an ——— function of x:
(a) Constant (b) Implicit
(c) Explicit (d) Inverse

Q.42 $y = \log_{10} x$ is known as the ——— of x:
(a) Common logarithmic
(b) Natural logarithmic
(c) Exponential
(d) None of these

Q.43 A function $I : X \to X$ for any set X, of the form $I(x) = x \; \forall \; x \in X$ is called a/an ——— function:
(a) Constant (b) Implicit
(c) Identity (d) Inverse

Q.44 $x^2 + xy + y^2 = 2$ is a/an ——— function of x: *(Faisalabad Board 2019)*
(a) Constant (b) Inverse
(c) Explicit (d) Implicit

Q.45 $\cos h^2 x - \sin h^2 x =$ ——— *(Lahore Board 2018 GI)*
(a) 1 (b) −1
(c) 0 (d) 2

Q.46 If $x = at^2$ and $y = 2at$ all equations of a curve, then t is called. *(Bahawalpur Board 2018)(Sahiwal Board 2019)*
(a) Variable (b) Constant
(c) Parameter (d) Co-efficient

Q.47 f(x) ≡ sin x is a/an ——— function:
(a) Constant (b) Odd
(c) Even (d) None of these

Q.48 If $x = e^y$, then $y = \log_e x = \ln x$, is known as the ——— of x:
(a) Common logarithmic
(b) Natural logarithmic
(c) Exponential
(d) None of these

Q.49 A function f from a set X to a set Y, then set of corresponding elements y in Y is called the ——— of f:
(a) Range (b) Domain
(c) Co-domain (d) None of these

Q.50 In y = f(x), the variable y is called ——— variable of a function f:
(a) Dependent (b) Independent
(c) Pre-image of x (d) None of these

Q.51 The term function was recognized by German Mathematician ———
(a) Euler (b) Lagrange
(c) Newton (d) Leibniz

Q.52 Let X and Y be the set of real numbers, a function $C : X \to Y$ defined by $C(x) = a \ \forall \ x \in X$, $a \in Y$ and a is a constant number. Then C is called a/an ——— function:
(a) Constant (b) Implicit
(c) Identity (d) Inverse

Q.53 A function, in which the variable appears as exponent (power), is called a/an ——— function:
(a) Constant (b) Explicit
(c) Exponential (d) Inverse

Q.54 If x and y are so mixed up and y cannot be expressed in terms of the independent variable x, then y is called a/an ——— function of x:
(a) Constant (b) Explicit
(c) Implicit (d) Inverse

Q.55 $\sin h^{-1} x =$
(a) $ln\left(x+\sqrt{x^2-1}\right)$ (b) $ln\left(x+\sqrt{x^2+1}\right)$
(c) $ln\left(1+\sqrt{x^2-1}\right)$ (d) $ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$

Q.56 $\cos h^{-1} x =$
(a) $ln\left(1+\sqrt{x^2-1}\right)$ (b) $ln\left(x+\sqrt{x^2+1}\right)$
(c) $ln\left(x+\sqrt{x^2-1}\right)$ (d) $ln\left(\frac{x+1}{x-1}\right)$

Q.57 $\tan h^{-1} x =$
(a) $\frac{1}{2} ln\left(\frac{1+x}{1-x}\right)$ (b) $\frac{1}{2} ln\left(\frac{x+1}{x-1}\right)$
(c) $ln\left(\frac{1+\sqrt{1-x^2}}{2}\right)$ (d) $ln\left(\frac{1+\sqrt{x^2+1}}{2}\right)$

Q.58 $\sec h^{-1} x =$
(a) $\frac{1}{2} ln\left(\frac{x+1}{x-1}\right)$ (b) $ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)$
(c) $ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ (d) $ln\left(\frac{1+x}{1-x}\right)$

Q.59 $\csc h^{-1} x =$
(a) $ln\left(\frac{1+\sqrt{x^2+1}}{x}\right)$ (b) $ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$
(c) $ln\left(\frac{1+\sqrt{1+x^2}}{2}\right)$ (d) $\frac{1}{2} ln\left(\frac{x+1}{x-1}\right)$

Q.60 $\cot h^{-1} x =$
(a) $ln\left(\frac{x+2}{x-2}\right)$ (b) $\frac{1}{2} ln\left(\frac{1+x}{1-x}\right)$
(c) $\frac{1}{2} ln\left(\frac{x+1}{x}\right)$ (d) $\frac{1}{2} ln\left(\frac{x+1}{x-1}\right)$

Q.61 Which of the following function is a polynomial function?
(a) $\frac{x^3-1}{x+2}, x \neq -2$ (b) $x^5+6x^4+7x^3+x^2+\sqrt{x}+4$
(c) $\frac{2x^2+7x+4}{10}$ (d) $ax^2 + b\sqrt{x} + c$

Q.62 Which one is a constant function?
(a) $f(x) = 2x^2$ (b) $f(x) = x$
(c) $f(x) = 3x + 1$ (d) $f(x) = 14$

Q.63 Which one is an identity function?
(a) $f(x) = x^2$ (b) $f(x) = g(x)$
(c) $f(x) = x$ (d) $f(x) = 1$

Q.64 Which one is not an exponential function?
(a) 2^x (b) n^x
(c) $e^{x/2}$ (d) x^n

Q.65 Which one is an exponential function?
(a) 2^x (b) x^2
(c) $\log_2 x$ (d) x^e

Q.66 If f(x) = ax + b, where $a \neq 0$, b are real, then f(x) is a:
(a) Constant function (b) Absolute function
(c) Linear function (d) Quadratic function

Q.67 If f(x) = |x|, f(x) is a:
(a) Constant function (b) Absolute function
(c) Linear function (d) Quadratic function

Q.68 If $f(x) = x^n$, where n is a positive integer, then f(x) is a:
(a) Constant function (b) Absolute function
(c) Linear function (d) Power function

Q.69 sin h x =
(a) $\frac{1}{2}(e^x - e^{-x})$ (b) $\frac{1}{2}(e^x + e^{-x})$
(c) $e^x - e^{-x}$ (d) $e^x + e^{-x}$

Q.70 cos h x =
(a) $\frac{1}{2}(e^x - e^{-x})$ (b) $\frac{1}{2}(e^x + e^{-x})$
(c) $\frac{1}{4}(e^x - e^{-x})$ (d) $\frac{1}{4}(e^x + e^{-x})$

Q.71 tan h x = *(Rawalpindi Board 2017 G-II)*
(a) $\frac{e^x + e^{-x}}{e^x - e^{-x}}$ (b) $\frac{1}{2}(e^x + e^{-x})$
(c) $\frac{2}{e^x + e^{-x}}$ (d) $\frac{e^x - e^{-x}}{e^x + e^{-x}}$

Q.72 If y is an image of x under the function f, we denote it by:
(a) x = f(y) (b) x = y
(c) y = f(x) (d) f(x, y) = c

Q.73 Inverse of $f(x) = \sqrt{x+1}$ is:
(a) $f^{-1}(x) = x^2 - 1$ (b) $f^{-1}(x) = \frac{1}{\sqrt{x+1}}$
(c) $f^{-1}(x) = 1 - x^2$ (d) $f^{-1}(x) = x^2 + 1$

Q.74 If f(−x) = −f(x) for all x in the domain of f, then f is:
(a) Linear function (b) Identity function
(c) Odd function (d) Even function

Q.75 If f is any function, then $\frac{f(x) + f(-x)}{2}$ is always:
(a) Even (b) Odd
(c) One-one (d) Zero

Q.76 Let f(x) = cos x, then f(x) is an:
(a) Even function (b) Odd function
(c) Power function (d) None of these

Q.77 Let $f(x) = x^3 + \sin x$, then f(x) is: *(D.G. Khan Board 2015 G-II)*
(a) Even function (b) Odd function
(c) Power function (d) None of these

Q.78 Let $f(x) = x^3 + \cos x$, then f(x) is:
(a) An odd function (b) An even function
(c) Neither even nor odd (d) A constant function

Q.79 Parametric equations x = a cos t, y = a sin t represent the equation of: *(Bahawalpur Board 2018)*
(a) Line (b) Circle
(c) Parabola (d) Ellipse

Q.80 Parametric equations: x = a cos θ, y = b sin θ represent the equation of:
(a) Parabola (b) Hyperbola
(c) Ellipse (d) Circle

Q.81 Parametric equations x = a sec θ, y = b tan θ represent the equation of: *(Faisalabad Board 2018)*
(a) Line (b) Parabola
(c) Ellipse (d) Hyperbola

Q.82 $\cos h^2 x - \sin h^2 x =$ *(Gujranwala Board 2014)*
(a) −1 (b) 1
(c) 2 (d) 0

Q.83 $\cos h^{-1} x =$ *(Lahore Board 2014 G-II)*
(a) $ln\left(x + \sqrt{x^2+1}\right)$ (b) $ln\left(x + \sqrt{x^2-1}\right)$
(c) $\frac{1}{2} ln\left|\frac{1+x}{1-x}\right|$ (d) $\frac{1}{2} ln\left|\frac{x+1}{x-1}\right|$

Q.84 x = 3 cos t, y = 3 sin t represents: *(Lahore Board 2014 G-II)*
(a) Line (b) Circle
(c) Parabola (d) Hyperbola

Q.85 $\cos h^2 x + \sin h^2 x =$ _____ *(Gujranwala Board 2013)*
(a) 1 (b) cos h2x
(c) sin h2x (d) 2 cos h2x

Q.86 A function f : X → Y defined by f(x) = x is called: *(Gujranwala Board 2012)*
(a) Identity function (b) Constant function
(c) Linear function (d) Odd function

Q.87 x = a cos t, y = a sin t are parametric equations of: *(Lahore Board 2012 G-II)*
(a) Parabola (b) Circle
(c) Ellipse (d) Hyperbola

Q.88 $\sec^2 x =$
(a) $\cos h^2 x$ (b) $1 + \tan h^2 x$
(c) $1 - \tan h^2 x$ (d) $\tan h^2 x - 1$

Q.89 If f(x) = x sec x, then f(0): *(Faisalabad Board 2016)*
(a) −1 (b) 1
(c) 0 (d) ∞

Q.90 The expression $ln\left(x + \sqrt{x^2+1}\right)$ equals: *(Lahore Board 2016 G-I)*
(a) $\sin h^{-1} x$ (b) $\cos h^{-1} x$
(c) $\tan h^{-1} x$ (d) $\operatorname{cosec} h^{-1} x$

Q.91 f(x) = x cot x is: *(Sargodha Board 2016)*
(a) Linear function (b) Quadratic function
(c) Even function (d) Odd function

Q.92 $2 \sinh x =$ *(Lahore Board 2017 G-I)*

(a) $e^x - e^{-x}$ (b) $e^x + e^{-x}$

(c) $\frac{e^x - e^{-x}}{2}$ (d) $\frac{e^x + e^{-x}}{2}$

Q.93 $\frac{e^x - e^{-x}}{2} =$ *(Gujranwala Board 2018)*

(a) Sin x (b) Cos x

(c) Sinh x (d) Cosh x

Q.94 A function I: $X \to X$ of the form $I(x) = x$, $\forall x \in X$ is called. *(Sargodha Board 2017)*

(a) Linear function (b) Constant function

(c) Identity function (d) Even function

Q.95 $f(x) = x^{2/3}$ is a/an: *(Gujranwala Board 2014)*

(a) Even function

(b) Odd function

(c) Neither even nor odd

(d) Cubic function

Q.96 Let $f(x) = x^2 + \cos x$, then f(x). *(Lahore Board 2018 G-I)*

(a) Odd function (b) Constant function

(c) Even function (d) Neither even nor odd

Q.97 If at least one vertical line meets the curve at more than two points then curve is. *(Lahore Board 2018 G-II)*

(a) A function (b) Not a function

(c) One to one function (d) Onto function

Q.98 The linear function $f(x) = ax + b$ becomes identity function if. *(Multan Board 2018 G-I)*

(a) $a = 0$, $b = 1$ (b) $a = 1$, $b = 0$

(c) $a = 0$, $b = 0$ (d) $a = 1$, $b = 1$

Q.99 $\ln\left(\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right) =$, $x \neq 0$ *(Sahiwal Board 2018)*

(a) $\cosh^{-1} x$ (b) $\tanh^{-1} x$

(c) $\text{sech}^{-1} x$ (d) $\text{cosech}^{-1} x$

Q.100 The linear function $f(x) = ax + b$ becomes constant if. *(Sargodha Board 2018)*

(a) $a = 10$, $b = 0$ (b) $a = 0$, $b = 1$

(c) $x = 1$, $b = 0$ (d) $x = 10$, $b = 0$

Q.101 Cosh 2x = : *(Faisalabad Board 2019 G-II)*

(a) $\frac{e^{2x} - e^{-2x}}{2}$ (b) $\frac{e^{2x} + e^{-2x}}{2}$

(c) $\frac{e^{-2x} - e^{-2x}}{2}$ (d) $\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$

Q.102 The range of $f(x) = x^2$ is *(Gujranwala Board 2019 G-I)*

(a) $(-\infty, 0)$ (b) $(-\infty, \infty)$

(c) $(-1, 0)$ (d) $(0, \infty)$

Q.103 The function $y = 27 + x^2$ is a / an: *(Lahore Board 2019)*

(a) Constant function (b) Even function

(c) Implicit function (d) Explicit function

Q.104 If $f(x) = x^3 + x$, then $f(x)$ is: *(Multan Board 2019)*

(a) Constant function (b) Even function

(c) Odd function (d) Implicit function

Q.105 $x = 3 \cos t$, $y = 3 \sin t$ represents: *(Multan Board 2019)*

(a) Line (b) Circle

(c) Ellipse (d) Hyperbola

Q.106 $x = at^2$ and $y = -2at$ are parametric equations of the curve. *(Sargodha Board 2019)*

(a) $y^2 = -4ax$ (b) $y^2 = 4ax$

(c) $x^2 = -4ay$ (d) $x^2 = 4ay$

EXERCISE 1.2

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find fog (x), when

$$f(x) = \sqrt{x+1}\ ;\ g(x) = \frac{1}{x^2}\ ,\ x \neq 0$$

(Bahawalpur Board 2016)(Multan Board 2017 G-I)

Sol: $\text{fog}(x) = f(g(x))$

$$= f\left(\frac{1}{x^2}\right) = \sqrt{\frac{1}{x^2} + 1} = \sqrt{\frac{1+x^2}{x^2}} = \frac{\sqrt{1+x^2}}{|x|}$$

Q.2 Find fof(x) if $f(x) = \sqrt{x+1}$

(Lahore Board 2016 G-I)

Sol: $\text{fof}(x) = f(f(x)) = f\left(\sqrt{x+1}\right) = \sqrt{\sqrt{x+1}+1}$

Q.3 If $g(x) = \frac{1}{x^2}$, find g o g(x).

(Multan Board 2017 G-I)

Sol: $g \circ g(x) = g(g(x))$

$$= g\left(\frac{1}{x^2}\right) = \frac{1}{\left(\frac{1}{x^2}\right)^2} = \frac{1}{\frac{1}{x^4}} = x^4$$

Q.4 Find f o g (x) if $f(x) = \frac{1}{\sqrt{x-1}}$, $g(x) = \frac{1}{x}$.

(Sahiwal Board 2018)

Sol: $f \circ g(x)$

$$= f(g(x))$$

$$= f\left(\frac{1}{x^2}\right)$$

$$= \frac{1}{\sqrt{\frac{1}{x^2} - 1}} = \frac{1}{\sqrt{\frac{1-x^2}{x^2}}} = \frac{1}{\frac{\sqrt{1-x^2}}{|x|}} = \frac{|x|}{\sqrt{1-x^2}}$$

Q.5 Find $f^{-1}(x)$; $f(x) = (-x+9)^3$

(Lahore Board 2010, 2017 G-II)(Bahawalpur Board 2014)

Sol: See Long Question 1

Q.6 Without finding the inverse, state the domain and range of f^{-1}, where

$f(x) = 2+\sqrt{x-1}$ **.** *(Lahore Board 2015 G-I)*

Sol: For domain, we must have $x - 1 \geq 0 \Rightarrow x \geq 1$

$\Rightarrow$ Domain $f = D_f = [1, +\infty[$

For range, $x - 1 \geq 0 \Rightarrow \sqrt{x-1} \geq 0 \Rightarrow 2 + \sqrt{x-1} \geq 2$

$\Rightarrow f(x) \geq 2$

Range $f = R_f = [2, +\infty[$

By definition of inverse function f^{-1}, we have

Df^{-1} = domain f^{-1} = range $f = [2, +\infty[$

Rf^{-1} = range f^{-1} = domain $f = [1, +\infty[$

Q.7 Without finding $f^{-1}(x)$, state domain and range of $f^{-1}(x)$ if $f(x) = \sqrt{x+2}$

(Bahawalpur Board 2018)

Ans. $f(x) = \sqrt{x+2}$

Dom $f = [-2, \infty[$

range $f = [0, \infty[$

Dom f^{-1} = Range $f = [0, \infty[$

Range f^{-1} = Dam $f = [-2, \infty[$

Q.8 If $f(x) = \sqrt{x+1}$ and $g(x) = \frac{1}{x^2}$, find g of (x).

(Lahore Board 2017 G-II)

Sol: g of (x) $= g(f(x))$

$= g(\sqrt{x+1}) = \frac{1}{(\sqrt{x+1})^2} = \frac{1}{x+1}$

Q.9 For the real valued function f defined as

$f(x) = \frac{2x+1}{x-1}$, find $f^{-1}(x)$ and find $f^{-1}(-1)$.

(Gujranwala Board 2019)

Sol: See Long Question 2

Q.10 $f(x) = 3x^4 - 2x^2$, $g(x) = \frac{2}{\sqrt{x}}$, find f(g(x)).

(Lahore Board 2019 G-II)

Sol: $fog(x) = f[g(x)] = 3[g(x)]^4 - 2(g(x))^2$

$= 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left[\frac{2}{\sqrt{x}}\right]^2 = 3\left(\frac{16}{x^2}\right) - 2\left(\frac{4}{x}\right)$

$= \frac{48}{x^2} - \frac{8}{x} = \frac{48-8x}{x^2} = \frac{8(6-x)}{x^2}$

LONG QUESTIONS

Q.1 For the real valued function f, defined below, find (i) $f^{-1}(x)$ (ii) $f^{-1}(-1)$ and verify $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ where $f(x) = (-x+9)^3$

(Bahawalpur Board 2014)(Sargodha Board 2016)

Sol: (a) Let $y = f(x) \Rightarrow y = (-x+9)^3$

Solving the equation for x

$\frac{1}{y^3} = -x + 9 \Rightarrow x = 9 - \frac{1}{y^3}$

$f^{-1}(y) = 9 - \frac{1}{y^3}$ $[\Theta\ x = f^{-1}(y)]$

To find $f^{-1}(x)$, replace y by x.

$f^{-1}(x) = 9 - \frac{1}{x^3}$

$f(f^{-1}(x)) = f(a - x^{\frac{1}{3}}) = (-(a - x^{\frac{1}{3}}) + 9)^3$

$= (-9 + x^{\frac{1}{3}} + 9)^3 = (x^{\frac{1}{3}})^3 = x$

$f^{-1}(f(x)) = f^{-1}(-x+9)^3 = 9 - [(-x+9)^3]^{\frac{1}{3}}$

$= 9 - (-x+9) = 9 + x - 9 = x$

Q.2 For the real valued function f defined as

$f(x) = \frac{2x+1}{x-1}$, find $f^{-1}(x)$ and find $f^{-1}(-1)$ and verify. $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(Lahore Board 2019)

Sol: Let $y = f(x) = \frac{2x+1}{x-1}$

$y(x-1) = 2x + 1$

$xy - y = 2x + 1$

$xy - 2x = 1 + y$

$x(y-2) = 1 + y$

$x = \frac{1+y}{y-2} = f^{-1}(y)$

Replace y by x

$f^{-1}(x) = \frac{1+x}{x-2}$

$f(f^{-1}(x))$

$= f\left(\frac{1+x}{x-2}\right) = \frac{2\left(\frac{1+x}{x-2}\right)+1}{\frac{1+x}{x-2}-1} = \frac{\frac{2+2x+x-2}{x-2}}{\frac{1+x-x+2}{x-2}} = \frac{3x}{3} = x$

$f^{-1}(f(x))$

$= f^{-1}\left(\frac{2x+1}{x-1}\right) = \frac{1+\frac{2x+1}{x-1}}{\frac{2x+1}{x-1}-2} = \frac{\frac{x-1+2x+1}{x-1}}{\frac{2x+1-2x+2}{x-1}} = \frac{3x}{3} = x$

$f^{-1}(-1) = \frac{1+(-1)}{-1-2} = \frac{0}{-3} = 0$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 If $f(x) = \sqrt{x+1}$, domain of f equal to:
(Sahiwal Board 2014)
(a) $(1, \infty)$ (b) $(-1, -\infty)$
(c) $[-1, \infty)$ (d) $(-\infty, \infty)$

Q.2 If $f(x) = 2x + 1$, then f of (x) is equal to:
(Rawalpindi Board 2013)(D.G.K Board 2013 G-I)
(a) $4x + 1$ (b) $4x + 3$
(c) $4x - 3$ (d) $4x - 1$

Q.3 The range of the function $f(x) = \frac{|x+2|}{x+2}$, $x \neq -2$ is:
(D.G.K Board 2015 G-II)
(a) $\{1\}$ (b) $\{-1, 1\}$
(c) $\{-1\}$ (d) $R - \{-1, 1\}$

Q.4 The range of $f(x) = x^2$ is: *(D.G.K Board 2015 G-I)*
(a) $(-\infty, \infty)$ (b) $(-\infty, 0)$
(c) $[0, \infty)$ (d) $(-1, 0)$

Q.5 Range of $f(x)=(x-5)^2$ is: *(D.G.K Board 2012 G-I)*
(a) $[5, \infty)$ (b) $(5, \infty)$
(c) $[0, \infty)$ (d) $(-\infty, \infty)$

Q.6 Let $f(x) = 4 - x$, $g(x) = 2x + 1$, then gof (x) =
(a) $5 - x$ (b) $7 - 2x$
(c) $9 - 2x$ (d) $9 + 2x$

Q.7 Let $f(x) = 4 - x$, $g(x) = -2$, then fog (x) =
(a) 2 (b) 6
(c) 8 (d) 5

Q.8 Let $f(x) = \sqrt{x+4}$, $g(x) = 2x + 1$, then fog (x) =
(a) $\sqrt{2x-3}$ (b) $\sqrt{2x+5}$
(c) $\sqrt{2x-5}$ (d) $2x + 5$

Q.9 Let $f(x) = 4 - x$, then $f^2(x)$ =
(a) x (b) $-x$
(c) $4 + x$ (d) $x - 4$

Q.10 Let $f(x) = \sqrt{x+4}$, $g(x) = 4 - x$, then fog (x) =
(a) $\sqrt{x}$ (b) $\sqrt{16-x}$
(c) $\sqrt{8-x}$ (d) $2\sqrt{x+4}+1$

Q.11 Let $f(x) = 4 - x$, $h(x) = \sqrt{x+4}$, then foh (x) =
(a) $\sqrt{4-x}$ (b) $\sqrt{x}$
(c) $4 - \sqrt{4-x}$ (d) $4 - \sqrt{x+4}$

Q.12 If $f(x) = \frac{x}{x-1}$, $x \neq 1$ then $f^{-1}(x)$ equals:
(a) $\frac{x}{x-1}$ (b) $\frac{x-1}{x}$
(c) $\frac{x}{1-x}$ (d) $\frac{1-x}{x}$

Q.13 $\lim_{x \to 2} (3x^2 + 7x - 1) =$
(a) 20 (b) 21
(c) 25 (d) 26

Q.14 If $f(x) = 4x - 1$. Then f(1) = _____
(a) 3 (b) 2
(c) -1 (d) 7

Q.15 Let $f(x) = \sqrt{9 - x^2}$, then range of f is the set:
(a) $]-\infty, \infty[$ (b) $[0, \infty)$
(c) $[3, \infty]$ (d) $[-3, 3]$

Q.16 Let $f(x) = \sqrt{x^2 - 4}$, then domain of f is the set:
(a) $\{x \in R : |x| \geq 2\}$ (b) $\{x \in N : |x| \leq 2\}$
(c) $\{x \in R : |x| \leq 2\}$ (d) $\{x \in I : |x| \geq 2\}$

Q.17 Let $f(x) = 4 - x$, $g(x) = 2x + 1$, then fog(x) =
(a) $5 + 2x$ (b) $3 - 2x$
(c) $2 + 3x$ (d) $2 - 3x$

Q.18 If $f(x) = x^2 + 1$, then the value of fof is equal to:
(a) $x^4 + 2x^2 + 1$ (b) $x^4 - 2x^2 + 2$
(c) $x^4 + 2x^2 + 2$ (d) $x^4 + 2x^2 - 2$

Q.19 Let $f(x) = 4 - x$, $g(x) = 2x + 1$, then gof (x) =
(a) $5 - x$ (b) $7 - 2x$
(c) $9 - 2x$ (d) $9 + 2x$

Q.20 If $f(x) = \sqrt{x+1}$, then domain of f(x) is:
(a) $[0, \infty)$ (b) $[-1, \infty)$
(c) $[1, \infty]$ (d) $[1, \infty)$

Q.21 If $f(x) = \sqrt{x+1}$, then range of f(x) is:
(a) $(-\infty, \infty)$ (b) $[-\infty, \infty]$
(c) $[0, \infty)$ (d) $[-1, \infty)$

Q.22 If $f(x) = x^2 - x$, then f(−2) is equal to:
(Lahore Board 2015 G-I)
(a) 2 (b) 6
(c) 0 (d) −6

Q.23 If $g(x) = \frac{1}{x^2}$, $(x \neq 0)$, then gog(x) is equal to:
(Rawalpindi Board 2016)
(a) 1 (b) x^2
(c) x^4 (d) $\frac{1}{x^4}$

Q.24 If $f(x) = \frac{1}{x}$, then $f^{-1}(x)$ = *(Multan Board 2017 G-I)*
(a) x (b) $\frac{1}{x}$
(c) 1 (d) Does not exist

Q.25 Domain f^{-1} = Range f and Range f^{-1} = _____
(a) Domain f (b) Range f
(c) Domain f^{-1} (d) None of these

Q.26 If $f(x) = x^2$, then domain of f is.
(Sahiwal Board 2017)
(a) real number (b) Integer
(c) rational number (d) irrational number

Q.27 If $f(x) = x^2$, then range of f is.
(A.J.K Board 2017)
(a) All non negative real numbers
(b) Rational number
(c) Integer (d) Irrational number

Q.28 Range of $f(x) = x^2 + 1$ is
(Gujranwala Board 2018)
(a) R (b) $R - \{1\}$
(c) $R - \{-1\}$ (d) $[1, \infty)$

Q.29 Domain of $f(x) = x^2 + 1$ is.
(Lahore Board 2018 G-II)
(a) R (b) $R - \{1\}$
(c) $R - \{-1\}$ (d) $[1, \infty)$

Q.30 If $f(x) = \sqrt{x+4}$, then $f(x^2 + 4)$ is equal to.
(Rawalpindi Board 2018)
(a) $x^2 - 8$ (b) $\sqrt{x^2 - 8}$
(c) $\sqrt{x^2 + 8}$ (d) $x^2 + 8$

Q.31 If $g(x) = \frac{3}{x-1}$, then gog(4) =
(Multan Board 2018 G-II)
(a) 3 (b) 1
(c) undefined (d) 0

Q.32 The range of $f(x) = 2 + \sqrt{x-1}$ is.
(Sahiwal, Rawalpindi Board 2018)
(a) $[-1, \infty[$ (b) $[0, \infty)$
(c) $[2, \infty)$ (d) $(-2, \infty)$

Q.33 If $f(x) = -2x + 8$, then $f^{-1}(-1) =$
(Sargodha Board 2018)
(a) $\frac{9}{2}$ (b) $\frac{2}{9}$
(c) $\frac{7}{2}$ (d) $\frac{2}{7}$

Q.34 If $f(x) = 2x - 8$, then $f^{-1}(x) =$:
(Faisalabad Board 2019)
(a) $8 - 2x$ (b) $8 + 2x$
(c) $\frac{x+8}{2}$ (d) $\frac{x-8}{2}$

Q.35 If $f(x) = \cos x$, then $f^{-1}\left(\frac{\pi}{2}\right)$
(Gujranwala Board 2019 G-I)
(a) -1 (b) $-\frac{1}{2}$
(c) 0 (d) 1

Q.36 If $f(x) = \frac{1}{x^2}$ $(x \neq 0)$, then of fof(x) is
(Gujranwala Board 2019 G-II)
(a) x^4 (b) x^2
(c) 1 (d) $\frac{1}{x^4}$

Q.37 If $f(x) = 2x + 1$, then $f^{-1}(x) =$?:
(Lahore Board 2019)
(a) $2x - 1$ (b) $1 - 2x$
(c) $x - \frac{1}{2}$ (d) $\frac{x-1}{2}$

Q.38 The domain of $g(x) = 2x - 5$ is:
(Rawalpindi Board 2019)
(a) IR (b) the set of positive No.
(c) The set of negative real No.
(d) The set of non-negative real No.

EXERCISE 1.3

SHORT ANSWERS TO THE QUESTIONS

Q.1 Evaluate $\lim_{x\to-\infty} \frac{2-3x}{\sqrt{3+4x^2}}$
(Rawalpindi Board 2014)

Sol:

$$\lim_{x\to-\infty} \frac{2-3x}{\sqrt{3+4x^2}} = \lim_{x\to-\infty} \frac{-3\left(-\frac{2}{x}+3\right)}{-x\sqrt{\frac{3}{x^2}+4}} \quad \left[\sqrt{x^2} = |x| = -x \text{ as } x<0\right]$$

$$= \lim_{x\to-\infty} \frac{-\frac{2}{x}+3}{\sqrt{\frac{3}{x^2}+4}} = \frac{-0+3}{\sqrt{0+2}} = \frac{3}{2}$$

Q.2 Evaluate $\lim_{x\to+\infty} \frac{5x^4-10x^2+1}{-3x^3+10x^2+50}$

Sol:

$$\lim_{x\to+\infty} \frac{5x^4-10x^2+1}{-3x^3+10x^2+50} = \lim_{x\to+\infty} \frac{x^3\left(5x-\frac{10}{x}+\frac{1}{x^3}\right)}{x^3\left(-3+\frac{10}{x}+\frac{50}{x^3}\right)}$$

$$= \lim_{x\to+\infty} \frac{5x-\frac{10}{x}+\frac{1}{x^3}}{-3+\frac{10}{x}+\frac{50}{x^3}} = \frac{\infty-0+0}{-3+0+0} = \infty$$

Q.3 Evaluate $\lim_{x\to-1} \frac{x^3-x}{x+1}$ *(Sahiwal Board 2014 G-II)*
(Gujranwala Board 2018, 2019)

Sol:

$$\lim_{x\to-1} \frac{x^3-x}{x+1} = \lim_{x\to-1} \frac{x(x^2-1)}{x+1}$$

$$= \lim_{x\to-1} \frac{x(x+1)(x-1)}{x+1}$$

$$= \lim_{x\to-1} x(x-1) \quad [\Theta\, x \neq -1]$$

$$= (-1)(-1-1) = (-1)(-2) = 2$$

Q.4 Evaluate $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^2 - x - 6}$

(Multan Board 2013 G-II)

Sol: $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^2 - x - 6} = \dfrac{(2)^3 - 8}{(2)^2 - 2 - 6} = \dfrac{8-8}{4-8} = \dfrac{0}{-4} = 0$

Q.5 Evaluate $\lim\limits_{x \to 3} \dfrac{x-3}{\sqrt{x} - \sqrt{3}}$

(Rawalpindi Board 2013) (Sargodha Board 2016) (Lahore Board 2011 G-I)

Ans. $\lim\limits_{x \to 3} \dfrac{x-3}{\sqrt{x} - \sqrt{3}}$

$= \lim\limits_{x \to 3} \dfrac{(\sqrt{x})^2 - (\sqrt{3})^2}{\sqrt{x} - \sqrt{3}} = \lim\limits_{x \to 3} \dfrac{(\sqrt{x} - \sqrt{3})(\sqrt{x} + \sqrt{3})}{\sqrt{x} - \sqrt{3}}$

$= \lim\limits_{x \to 3} (\sqrt{x} + \sqrt{3}) = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$

Q.6 Evaluate $\lim\limits_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

(Lahore Board 2012 G-I)

Sol: By substituting x = 0, we have $\left(\dfrac{0}{0}\right)$ form, so we rationalize the numerator.

$\lim\limits_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} \times \dfrac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$

$= \lim\limits_{h \to 0} \dfrac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})}$

$= \lim\limits_{h \to 0} \dfrac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim\limits_{h \to 0} \dfrac{h}{h(\sqrt{x+h} + \sqrt{x})}$

$= \lim\limits_{h \to 0} \dfrac{1}{\sqrt{x+h} + \sqrt{x}} = \dfrac{1}{\sqrt{x+0} + \sqrt{x}} = \dfrac{1}{\sqrt{x} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

Q.7 Find the limit of $\lim\limits_{n \to \infty} \left(1 - \dfrac{1}{n}\right)^n$.

(Faisalabad, Sahiwal Board 2013) (Lahore Board 2014 G-I)

Ans. $\lim\limits_{x \to \infty} \left(1 - \dfrac{1}{n}\right)^n$

$= \lim\limits_{x \to \infty} \left(1 + \left(-\dfrac{1}{n}\right)^{n}\right)^{-1}$

$= \left(\lim\limits_{x \to \infty} \left(1 + \left(-\dfrac{1}{n}\right)^{-n}\right)^{-1}\right)$

$= e^{-1} = \dfrac{1}{e}$

Q.16 Express each limit in terms of e;

$\lim\limits_{n \to +\infty} \left(1 + \dfrac{1}{3n}\right)^n$

Sol: $\lim\limits_{n \to +\infty} \left(1 + \dfrac{1}{3n}\right)^n = \lim\limits_{n \to +\infty} \left[\left(1 + \dfrac{1}{3n}\right)^{3n}\right]^{\frac{1}{3}}$

$= \left[\lim\limits_{n \to +\infty} \left(1 + \dfrac{1}{3n}\right)^{3n}\right]^{\frac{1}{3}} = e^{\frac{1}{3}}$

Q.8 Express each limit in terms of e;

$\lim\limits_{x \to 0} (1 + 2x^2)^{\frac{1}{x^2}}$

Sol: $\lim\limits_{x \to 0} (1 + 2x^2)^{\frac{1}{x^2}} = \lim\limits_{x \to 0} (1 + 2x^2)^{\frac{2}{2x^2}}$

$= \lim\limits_{x \to 0} \left[(1 + 2x^2)^{\frac{1}{2x^2}}\right]^2$

$= \left[\lim\limits_{x \to 0} (1 + 2x^2)^{\frac{1}{2x^2}}\right]^2 = e^2$

Q.9 Express each limit in terms of e;

$\lim\limits_{x \to +\infty} \left(\dfrac{x}{1+x}\right)^x$

(Lahore Board 2011 G-I)(Gujranwala Board 2010)

Sol: $\lim\limits_{x \to +\infty} \left(\dfrac{x}{1+x}\right)^x = \lim\limits_{x \to +\infty} \left(\dfrac{1+x}{x}\right)^{-x}$

$= \lim\limits_{x \to +\infty} \left(\dfrac{1}{x} + \dfrac{x}{x}\right)^{-x}$

$= \lim\limits_{x \to +\infty} \left(\dfrac{1}{x} + 1\right)^{-x}$

$= \left[\lim\limits_{x \to +\infty} \left(1 + \dfrac{1}{x}\right)^x\right]^{-1} = [e]^{-1} = e^{-1}$

Q.10 Evaluate $\lim\limits_{h \to 0} (1 + 2h)^{\frac{\lambda t}{h}}$

(Rawalpindi Board 2017 G-II)

Sol: $\lim\limits_{h \to 0} (1 + 2h)^{\frac{\lambda t}{h}}$

$= \lim\limits_{h \to 0} \left[(1 + 2h)^{\frac{1}{2h}}\right]^{2\lambda t}$

$= \left[\lim\limits_{h \to 0} (1 + 2h)^{\frac{1}{2h}}\right]^{2\lambda t} = e^{2\lambda t}$

Q.11 Express $\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{2n}$ in terms of number "e"

(Multan Board 2018 G-II)

Sol: $\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{2n}$

$= \lim_{n\to\infty}\left[\left(1+\frac{3}{n}\right)^{n/3}\right]^6 = \left[\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{n/3}\right]^6$

$= e^6$

Q.12 Evaluate $\lim_{x\to+\infty}\frac{2-3x}{\sqrt{3+4x^2}}$

Sol: $\lim_{x\to+\infty}\frac{2-3x}{\sqrt{3+4x^2}}$

$= \lim_{x\to+\infty}\frac{x\left(\frac{2}{x}-3\right)}{x\sqrt{\frac{3}{x^2}+4}} \quad \left[\sqrt{x^2}=|x|=x \text{ as } x>0\right]$

$= \lim_{x\to+\infty}\frac{\frac{2}{x}-3}{\sqrt{\frac{3}{x^2}+4}} = \frac{0-3}{\sqrt{0+4}} = \frac{-3}{2}$

Q.13 Evaluate $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$

(Lahore Board 2015, 2017 G-I)

Sol: $\lim_{x\to 0}\frac{\sin ax}{\sin bx}\left(\frac{0}{0}\right)$ form

$= \lim_{x\to 0}\frac{a}{b}\frac{\sin ax}{ax}\times\frac{bx}{\sin bx} \quad (\because x\neq 0)$

$= \frac{a}{b}\lim_{x\to 0}\frac{\sin ax}{ax}\times\lim_{x\to 0}\frac{bx}{\sin bx} = \frac{a}{b}\times 1\times 1 = \frac{a}{b}$

Q.14 Evaluate $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$

(Faisalabad Board 2017)

Sol: $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}\left(\frac{0}{0}\right)$ form

$= \lim_{x\to 0}\frac{2\sin^2 x}{x^2} = 2\lim_{x\to 0}\frac{\sin^2 x}{x^2} = 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$

$= 2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = 2(1)^2 = 2$

Q.15 Evaluate $\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x}$

(Bahawalpur Board 2018)

Sol: $\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x}\left(\frac{0}{0}\right)$ form

$= \lim_{x\to 0}\frac{1-\cos x}{\sin^2 x}\times\frac{1+\cos x}{1+\cos x}$

$= \lim_{x\to 0}\frac{1-\cos^2 x}{\sin^2 x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{\sin^2 x(1+\cos x)}$

$= \lim_{x\to 0}\frac{1}{1+\cos x} = \frac{1}{1+\cos 0} = \frac{1}{1+1} = \frac{1}{2}$

Q.16 Evaluate $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$

(Lahore Board 2017 G-II)

Sol: $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$

$= \lim_{\theta\to 0}\frac{2\sin^2\frac{\theta}{2}}{\theta}$

$= \lim_{\theta\to 0}\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\times\lim_{\theta\to 0}\sin\frac{\theta}{2}$

$= (1)\times\sin 0 = (1)\times(0) = 0$

Q.17 Evaluate $\lim_{x\to 0}\frac{\sec x-\cos x}{x}$

(Lahore Board 2018 G-II)

Sol: $\lim_{x\to 0}\frac{\sec x-\cos x}{x}\left(\frac{0}{0}\right)$ form

$= \lim_{x\to 0}\frac{\frac{1}{\cos x}-\cos x}{x}$

$= \lim_{x\to 0}\frac{\frac{1-\cos^2 x}{\cos x}}{x}$

$= \lim_{x\to 0}\frac{\sin^2 x}{x\cos x}$

$= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\times\frac{x}{\cos x}$

$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\times\frac{\lim_{x\to 0} x}{\lim_{x\to 0}\cos x}$

$= (1)^2\times\frac{0}{\cos 0} = \frac{0}{1} = 0$

Q.18 Evaluate $\lim_{\theta \to 0} \dfrac{\tan\theta - \sin\theta}{\sin^3\theta}$

(Lahore Board 2012 G-II)

Sol: $\lim_{\theta \to 0} \dfrac{\tan\theta - \sin\theta}{\sin^3\theta}$ $\left(\dfrac{0}{0}\right)$ form

$$= \lim_{\theta \to 0} \frac{\dfrac{\sin\theta}{\cos\theta} - \sin\theta}{\sin^3\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta\left(\dfrac{1}{\cos\theta} - 1\right)}{\sin^3\theta}$$

$$= \lim_{\theta \to 0} \frac{\dfrac{1}{\cos\theta} - 1}{\sin^2\theta}$$

$$= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\cos\theta \sin^2\theta}$$

$$= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\cos\theta \sin^2\theta} \times \frac{1 + \cos\theta}{1 + \cos\theta}$$

$$= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\cos\theta \sin^2\theta (1 + \cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\sin^2\theta}{\cos\theta \sin^2\theta (1 + \cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{1}{\cos\theta (1 + \cos\theta)} = \frac{1}{\cos 0(1 + \cos 0)}$$

$$\Rightarrow \frac{1}{(1)(1+1)} = \frac{1}{2}$$

Q.19 Find $\lim_{x \to 0} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$, $x > 0$

(Faisalabad Board 2019 G-II)

Sol: $\lim_{x \to 0} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$, $x > 0$

Since $x > 0$ when $x \to 0$, then $\dfrac{1}{x} \to +\infty$, so

$$= \lim_{x \to 0} \frac{e^{1/x}\left[1 - \dfrac{1}{e^{1/x}}\right]}{e^{1/x}\left[1 + \dfrac{1}{e^{1/x}}\right]} = \lim_{x \to 0} \frac{\left[1 - \dfrac{1}{e^{\frac{1}{x}}}\right]}{\left[1 + \dfrac{1}{e^{\frac{1}{x}}}\right]}$$

$$= \frac{\lim_{x \to 0}\left[1 - \dfrac{1}{e^{\frac{1}{x}}}\right]}{\lim_{x \to 0}\left[1 + \dfrac{1}{e^{\frac{1}{x}}}\right]} = \frac{\left[1 - \dfrac{1}{e^{\frac{1}{0}}}\right]}{\left[1 + \dfrac{1}{e^{\frac{1}{0}}}\right]} = \frac{1 - \dfrac{1}{e^{\infty}}}{1 + \dfrac{1}{e^{\infty}}}$$

$$= \frac{1 - \dfrac{1}{\infty}}{1 + \dfrac{1}{\infty}} = \frac{1 - 0}{1 + 0} = 1$$

Q.20 Evaluate $\lim_{x \to 3} \dfrac{x - 3}{\sqrt{x} - \sqrt{3}}$. ***(Sahiwal Board 2019)***

Sol. $\lim_{x \to 3} \dfrac{x - 3}{\sqrt{x} - \sqrt{3}}$

$$= \lim_{x \to 3} \frac{(\sqrt{x})^2 - (\sqrt{3})^2}{\sqrt{x} - \sqrt{3}} = \lim_{x \to 3} \frac{(\sqrt{x} - \sqrt{3})(\sqrt{x} + \sqrt{3})}{\sqrt{x} - \sqrt{3}}$$

$$= \lim_{x \to 3} (\sqrt{x} + \sqrt{3}) = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$$

LONG QUESTIONS

Q.1 Evaluate $\lim_{x \to 0} \dfrac{1 - \cos 2x}{x^2}$. *(Rawalpindi Board 2016)*

Sol: See short question 29

Q.2 Evaluate $\lim_{x \to 0} \dfrac{\tan\theta - \sin\theta}{\sin^3\theta}$.

(Lahore Board 2013 G-II)(D.G.K Board 2014)
(Multan Board 2013 G-I, 2016 G-II)
(Rawalpindi Board 2013)(Sargodha Board 2017)

Sol: See short question 34

Q.3 Evaluate $\lim_{x \to 0} \dfrac{\sec x - \cos x}{x}$

(Gujranwala Board 2016)

Sol: See short question 33

Q.4 Find $\lim_{x \to 0} \dfrac{1 - \cos x}{\sin^2 x}$. *(Lahore Board 2017 G-I)*

Sol: See short question 30

Q.5 Evaluate $\lim_{\theta \to 0} \dfrac{1 - \cos p\theta}{1 - \cos q\theta}$.

(Gujranwala Board 2013) (Lahore Board 2016 G-I)
(Sahiwal Board 2014)(Rawalpindi Board 2017 G-II)
(D.G Khan Board 2017 G-II)

Sol: $\lim_{\theta \to 0} \dfrac{1 - \cos p\theta}{1 - \cos q\theta}$ $\left(\dfrac{0}{0}\right)$ form

$$= \lim_{\theta \to 0} \frac{2\sin^2\left(\dfrac{p\theta}{2}\right)}{2\sin^2\left(\dfrac{q\theta}{2}\right)} = \lim_{\theta \to 0} \left[\frac{\sin\left(\dfrac{p\theta}{2}\right)}{\sin\left(\dfrac{q\theta}{2}\right)}\right]^2$$

$$\Theta \qquad 1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

$$\Rightarrow \qquad 1-\cos p\theta = 2\sin^2 p\frac{\theta}{2}$$

$$1-\cos q\theta = 2\sin^2 q\frac{\theta}{2}$$

$$= \lim_{\theta\to 0}\left[\frac{\dfrac{\sin\left(\dfrac{p\theta}{2}\right)}{\dfrac{p\theta}{2}}\cdot\dfrac{p\theta}{2}}{\dfrac{\sin\left(\dfrac{q\theta}{2}\right)}{\dfrac{q\theta}{2}}\cdot\dfrac{q\theta}{2}}\right]^2 = \frac{p^2\left(\lim\limits_{\theta\to 0}\dfrac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}}\right)^2}{q^2\left(\lim\limits_{\theta\to 0}\dfrac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\right)^2}$$

$$= \frac{p^2(1)}{q^2(1)} = \frac{p^2}{q^2}$$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 $\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta}$ equals: *(Sahiwal Board 2014)*

(a) Zero (b) -1
(c) 1 (d) 2

Q.2 $\lim\limits_{x\to 0} e^{1/x} =$ _____, $x < 0$: *(D.G.K Board 2014 G-II)*

(a) -1 (b) 0
(c) 1 (d) ∞

Q.3 $\lim\limits_{x\to 0}\dfrac{e^x-1}{x} =$ _____ *(Multan Board 2014 G-II)*

(a) 0 (b) 1
(c) e (d) ∞

Q.4 $\lim\limits_{x\to 1}\dfrac{x^3-3x^2+3x-1}{x-1}$ equals: *(Bahawalpur Board 2014)(D.G.K Board 2011)*

(a) 0 (b) -1
(c) 1 (d) 3

Q.5 $\lim\limits_{x\to 0}\dfrac{x-2}{\sqrt{x}-\sqrt{2}} =$ _____ *(Bahawalpur Board 2014)(Multan Board 2017 G-I)(Sargodha Board 2017)*

(a) $\sqrt{2}$ (b) 2
(c) $3\sqrt{3}$ (d) 0

Q.6 $\lim\limits_{x\to 0}\dfrac{x^2}{\sin 7x \sin 5x}$. *(Sargodha Board 2013)*

(a) $\frac{7}{5}$ (b) $\frac{5}{7}$
(c) 2 (d) $\frac{1}{35}$

Q.7 $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}$ is equal to:

(a) $\ln x$ (b) $\ln a$
(c) 1 (d) 0

Q.8 $\lim\limits_{x\to\infty}\left(\frac{1}{2}\right)^x =$ *(Faisalabad Board 2013)*

(a) 0 (b) 1
(c) $\frac{1}{2}$ (d) ∞

Q.9 $\lim\limits_{x\to 0}\dfrac{\sin x}{x} =$ *(D.G.K Board 2015 G-I)*

(a) 1 (b) $\frac{\pi}{180}$
(c) -1 (d) $\frac{180}{\pi}$

Q.10 $\lim\limits_{x\to 0}(1+x)^{1/x}$ equals: *(D.G.K Board 2013 G-I)*

(a) e^{-1} (b) e
(c) e^2 (d) e^3

Q.11 $\lim\limits_{x\to -1}\dfrac{x^3-x}{x+1}$ equals: *(D.G.K Board 2010) (Bahawalpur Board 2019)*

(a) 0 (b) 1
(c) 2 (d) ∞

Q.12 $\lim\limits_{x\to 4}\dfrac{x-4}{\sqrt{x}-2}$ equals to: *(D.G.K Board 2010)*

(a) 0 (b) $2\sqrt{2}$
(c) 2 (d) 4

Q.13 $\lim\limits_{x\to 0}\dfrac{x}{\sin x^\circ}$ is equal to *(Gujranwala Board 2009)*

(a) $\frac{\pi}{180}$ (b) $\frac{180}{\pi}$
(c) 0 (d) 1

Q.14 $\lim\limits_{x\to 3}\dfrac{2x^2-5x-3}{3-x} =$

(a) -5 (b) -7
(c) -9 (d) Limit does not exist

Q.15 $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3} =$

(a) 5 (b) 1
(c) 2 (d) 6

Q.16 $\lim_{x\to 16} \frac{x-16}{\sqrt{x}-4} =$

(a) 3 (b) 18
(c) 8 (d) 12

Q.17 $\lim_{x\to 3} \frac{x^2+1}{x+3} =$

(a) $\frac{3}{5}$ (b) $\frac{1}{4}$
(c) $\frac{1}{2}$ (d) $\frac{5}{3}$

Q.18 $\lim_{x\to \pi} \frac{\sin x}{\pi - x} =$

(a) 0 (b) 1
(c) -1 (d) x

Q.19 $\lim_{x\to 1} \frac{x^2-16}{x-4} =$

(a) 8 (b) 14
(c) 10 (d) 12

Q.20 $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} =$

(a) 4 (b) $\frac{1}{2}$
(c) 1 (d) $\frac{1}{4}$

Q.21 $\lim_{x\to 2} \frac{x^3-8}{x-2} =$

(a) 10 (b) 13
(c) ∞ (d) 12

Q.22 $\lim_{x\to 3} \frac{x^2+2x}{2x+3} =$

(a) 5 (b) $\frac{4}{3}$
(c) $\frac{3}{5}$ (d) $\frac{5}{3}$

Q.23 $\lim_{x\to 3} \frac{x^2-5x+6}{x-3} =$

(a) 1 (b) 3
(c) $\frac{1}{4}$ (d) None of these

Q.24 $\lim_{x\to\infty} \frac{x^2+1}{2x^3-x} =$

(a) $\frac{1}{2}$ (b) 1
(c) 2 (d) 0

Q.25 $\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x} =$

(a) 2 (b) $\frac{1}{6}$
(c) $\frac{1}{2}$ (d) $\frac{1}{3}$

Q.26 $\lim_{x\to a} \frac{x^n-a^n}{x-a}$

(a) $n\,a^n$ (b) $n\,a^{n-1}$
(c) 0 (d) Does not exist

Q.27 $\lim_{x\to 2} (2x^2-5x+3) =$

(a) 1 (b) 2
(c) 3 (d) 4

Q.28 $\lim_{x\to -1} (2x^2+3x+1) =$

(a) 0 (b) 6
(c) 4 (d) 7

Q.29 $\lim_{x\to 0} \frac{x^4-a^4}{x-a}$ is: *(Gujranwala Board G-II)*

(a) $4a^3$ (b) $3a^4$
(c) 0 (d) Note defend

Q.30 $\lim_{x\to 1} \frac{x^2-1}{x^2+x-2} =$

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$
(c) $\frac{3}{2}$ (d) 1

Q.31 $\lim_{x\to 1} \frac{3-x}{x+2} =$

(a) 1 (b) $\frac{2}{3}$
(c) $\frac{3}{2}$ (d) 0

Q.32 $\lim_{x\to 2} \frac{\sqrt{x+7}-3}{x-2} =$

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$
(c) $\frac{1}{5}$ (d) $\frac{1}{2}$

Q.33 $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} =$

(a) 0 (b) $\frac{1}{4}$
(c) 1 (d) $\frac{1}{2}$

Q.34 $\lim_{x\to 0} \frac{\sin 7\theta}{7\theta} =$ *(Lahore Board 2011G-I)*

(a) 7 (b) $\frac{1}{7}$
(c) 1 (d) Zero

Q.35 $\lim_{x\to a} [f(x) - g(x)] =$ _____

(a) $\lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ (b) $\lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$
(c) $\lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ (d) $\lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

Q.36 $\lim_{x\to a}[f(x)\times g(x)] =$ ————

(a) $\lim_{x\to a} f(x)\times\lim_{x\to a} g(x)$ (b) $\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$

(c) $\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$ (d) $\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$

Q.37 $\lim_{x\to a}[f(x)]^n =$ ————

(a) $\lim_{x\to a} f(x)$ (b) $n\times\left[\lim_{x\to a} f(x)\right]$

(c) $\left[\lim_{x\to a} f(x)\right]^n$ (d) None of these

Q.38 If k is any real number, then $\lim_{x\to a}[k.f(x)] =$ ——

(a) $k\lim_{x\to a} f(x)$ (b) $k+\lim_{x\to a} f(x)$

(c) $\lim_{x\to a} f(x)$ (d) None of these

Q.39 $\lim_{x\to a}[f(x)+g(x)] =$ ————

(a) $\lim_{x\to a} f(x)\times\lim_{x\to a} g(x)$

(b) $\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$

(c) $\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$

(d) $\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$

Q.40 $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) =$ ————, $\lim_{x\to a} g(x)\neq 0$

(a) $\frac{f(x)}{g(x)}$ (b) $\frac{\lim_{x\to a} g(x)}{\lim_{x\to a} f(x)}$

(c) $\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (d) None of these

Q.41 Let $f(x) = x+\frac{1}{x}$, then $f\left(\frac{1}{x}\right) =$

(a) $f(x^2+1)$ (b) $f(x^2)$

(c) $\frac{1}{f(x)}$ (d) $f(x)$

Q.42 Let $f(x) = \frac{x}{x-2}$ then domain of f is the set of all real numbers except:

(a) 0 (b) 1

(c) 2 (d) 3

Q.43 $\lim_{\theta\to 0}\frac{1-\cos p\theta}{1+\cos q\theta}$ equals: *(Lahore Board 2014 G-I)*

(a) 0 (b) $\frac{p}{q}$

(c) $\frac{p^2}{q^2}$ (d) $\frac{q^2}{p^2}$

Q.44 $\frac{e^{2x}-1}{2e^x}$ is equal to: *(Lahore Board 2014 G-I)*

(a) sin x (b) cos x

(c) sin hx (d) cos hx

Q.45 $\lim_{x\to 0}\frac{e^{-x/2}}{1+e^{-x/2}} =$ *(Lahore Board 2013 G-II)*

(a) 0 (b) 1

(c) −1 (d) ∞

Q.46 $\lim_{x\to\infty}\left(1+\frac{x}{2}\right)^{1/x}$ equals: *(Lahore Board 2015 G-II)*

(a) e (b) e^{-1}

(c) e^2 (d) $\sqrt{e}$

Q.47 $\lim_{x\to 0}\frac{\sin 3x}{\sin 2x}$ is equal to: *(Lahore Board 2015 G-I)*

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$

(c) $\frac{1}{6}$ (d) $\frac{1}{4}$

Q.48 $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x =$ *(Bahawalpur Board 2016)*

(a) 4e (b) 3e

(c) 2e (d) e

Q.49 $\lim_{x\to 0}\frac{\sin x^0}{x^0} =$ *(Gujranwala Board 2016)*

(a) 1 (b) 0

(c) $\frac{\pi}{180}$ (d) $\frac{180}{\pi}$

Q.50 $\lim_{x\to 2}\frac{x^5-32}{x-2}$ equals: *(Lahore Board 2016 G-I)*

(a) 40 (b) 60

(c) 80 (d) 120

Q.51 $\lim_{x\to 0}\frac{\sin a\,x}{\sin b\,x}$ is equal to. *(Rawalpindi Board 2017 G-I)*

(a) $-\frac{a}{b}$ (b) $-\frac{b}{a}$

(c) $\frac{a}{b}$ (d) $\frac{b}{a}$

Q.52 $\lim_{x\to 0}\frac{x}{\tan x}$ is equal to. *(Rawalpindi Board 2017 G-II)*

(a) 0 (b) 1

(c) ∞ (d) $\frac{1}{2}$

Q.53 $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{2}}$ is equal to. *(Rawalpindi Board 2017 G-II, 2019)*

(a) e (b) $\sqrt{e}$

(c) 0 (d) $\frac{1}{e}$

Q.54 $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, where θ is measured is.

(Sahiwal Board 2017)

(a) degree (b) radian
(c) minutes (d) seconds

Q.55 $\lim_{x \to 2} \frac{\sqrt{x} - \sqrt{3}}{x - 3} =$ *(Sargodha Board 2019)*

(a) $\frac{1}{2\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\sqrt{3}$ (d) $2\sqrt{3}$

Q.56 $\lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) =$ *(D.G Khan Board 2017 G-I)*

(a) e (b) undefined
(c) $\log_e a$ (d) $\log_e e$

Q.57 $\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n$ *(Faisalabad Board 2017)*

(a) e (b) e^2
(c) e^{-1} (d) $\frac{1}{e^2}$

Q.58 $\lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta} =$

(Lahore Board 2017 G-I)(Sahiwal Board 2019)

(a) 0 (b) 1
(c) ∞ (d) $3\frac{0}{0}$

Q.59 If $\lim_{x \to 0} \frac{f(x) - 1}{x} = l$na, a > 0, then f(x):

(Lahore Board 2015 G-II)

(a) a^{-x} (b) a^x
(c) e^{-x} (d) e^x

Q.60 $\lim_{q \to 0} \frac{\sin 7\theta}{\theta}$ *(A.J.K Board 2017)*

(Multan Board 2018 G-II)(Rawalpindi Board 2018)

(a) 7 (b) $\frac{1}{7}$
(c) 1 (d) $\frac{2}{7}$

Q.61 $\lim_{h \to \infty} \left(1 + \frac{3}{n}\right)^{2n} =$ *(Faisalabad Board 2018)*

(a) e (b) e^2
(c) e^3 (d) e^6

Q.62 $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^{2n} =$:

(Faisalabad Board 2019 G-II)

(a) e (b) e^2
(c) e^4 (d) e^6

Q.63 $\lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^{2n} =$

(Gujranwala Board 2019 G-II)

(a) e (b) e^2
(c) e^n (d) zero

Q.64 $\lim_{x \to 4} \frac{x^2 - 6x + 8}{x - 4} =$ *(Multan Board 2019)*

(a) 4 (b) 2
(c) 6 (d) 8

EXERCISE 1.4

SHORT ANSWERS TO THE QUESTIONS

Q.1 **If** $f(x) = \begin{cases} x + 2 & x \leq 1 \\ c + 2 & x > -1 \end{cases}$

Find "c" so that $\lim_{x \to -1} f(x)$ **exists.**

(Gujranwala Board 2014) (Lahore Board 2018 G-I)

Sol.

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (c + 2) = c + 2$

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (x + 2)$

$= \lim_{x \to -1^-} (x) + \lim_{x \to -1^-} (2) = (-1) + 2 = -1 + 2 = 1$

As $\lim_{x \to -1} f(x)$ exist, therefore

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$

$\Rightarrow c + 2 = 1 \Rightarrow c = -1$

Q.2 **Discuss continuity of** $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ **at x = 3** *(Gujranwala Board 2016)*

(Rawalpindi Board 2019)

Sol: Given f(3) = 6

$\therefore$ the function is defined at x = 3.

Now $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

$= \lim_{x \to 3} \frac{(x + 3)(x - 3)}{x - 3}$

$= \lim_{x \to 3} (x + 3) = 3 + 3 = 6$

As $\lim_{x \to 3} f(x) = 6 = f(3)$

$\therefore$ f(x) is continuous at x = 3

Q.3 Discuss continuity of f(x) at x = 3 when

$$f(x) = \begin{cases} x-1, & \text{if } x < 3 \\ 2x+1, & \text{if } 3 \le x \end{cases}$$

(Lahore Board 2016 G-I)(Bahawalpur Board 2019)

Sol: $f(3) = 2(3) + 1 = 7$

$\therefore$ the function is defined at x = 3.

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (2x+1) = 2(3)+1 = 7$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} (x-1) = 3-1 = 2$$

Since, $\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$

$\therefore \lim_{x\to 3} f(x)$ does not exist.

Hence f(x) is not continuous at x = 3.

LONG QUESTIONS

Q.1 If $f(x) = \begin{cases} 3x & \text{if } x \le -2 \\ x^2 - 1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \ge 2 \end{cases}$ discuss continuity at x = 2 and x = −2.

(Lahore Board 2013, 2019 G-II)(Faisalabad Board 2013)
(Sahiwal Board 2015) (Multan Board 2012, 2014 G-I)

Sol: At x = 2

(i) $f(2) = 3$

(ii) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2} (x^2-1) = (2)^2 - 1$

$= 4 - 1 = 3$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2} (3) = 3$

Since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\therefore \lim_{x\to 2} f(x) = 3$

(iii) $\lim_{x\to 2} f(x) = f(2)$

Hence f(x) is continuous at x = 2.

Now at x = −2

(i) $f(-2) = 3(-2) = -6$

(ii) $\lim_{x\to -2^-} f(x) = \lim_{x\to -2} (3x) = 3(-2) = -6$

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2} (x^2-1) = (-2)^2 - 1$

$= 4 - 1 = 3$

Since $\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$

$\therefore \lim_{x\to -2} f(x)$ does not exist.

Hence f(x) is discontinuous at x = −2.

Q.2 If $F(x) = \begin{cases} \dfrac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} & x \neq 2 \\ k & x = 2 \end{cases}$ find the value of k so that f is continuous at x = 2.

(Lahore Board 2014 G-I, 2015 G-II, 2018 G-I)
(Sahiwal Board 2013)(Sargodha Board 2019)
(Multan Board 2016 G-I, 2018 G-II)
(Gujranwala Board 2012, 2019 G-II)
(Faisalabad Board 2017)

Sol: $f(2) = k$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \left(\frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \right)$$

$$= \lim_{x\to 2} \left(\frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \times \frac{\sqrt{2x+5} + \sqrt{x+7}}{\sqrt{2x+5} + \sqrt{x+7}} \right)$$

$$= \lim_{x\to 2} \left(\frac{(\sqrt{2x+5})^2 - (\sqrt{x+7})^2}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} \right)$$

$$= \lim_{x\to 2} \frac{(2x+5) - (x+7)}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})}$$

$$= \lim_{x\to 2} \frac{2x+5-x-7}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})}$$

$$= \lim_{x\to 2} \frac{x-2}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})}$$

$$= \lim_{x\to 2} \frac{1}{(\sqrt{2x+5} + \sqrt{x+7})} = \frac{1}{(\sqrt{2(2)+5} + \sqrt{2+7})}$$

$$= \frac{1}{\sqrt{9} + \sqrt{9}} = \frac{1}{(3+3)} = \frac{1}{6}$$

$$\lim_{x\to 2} f(x) = \frac{1}{6}$$

Since f is continuous at x = 2 therefore

$$\lim_{x\to 2} f(x) = f(2)$$

Q.3 Find the value of m and n so that given function f(x) is continues at x = 3, if

$$f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x + 9 & \text{if } x > 3 \end{cases}$$

(Gujranwala Board 2014, 2018)(Sahiwal Board 2018)
(Multan Board 2012 G-II, 2019 G-I)
(Lahore Board 2015 G-I)(Faisalabad Board 2019 G-I)

Sol: $f(3) = n$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} (mx) = 3m$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3} (-2x + 9)$$

$= -2(3) + 9 = -6 + 9 = 3$

Since f is continuous therefore.

$\lim_{x\to3} f(x) = f(3)$

$\Rightarrow \lim_{x\to3^-} f(x) = \lim_{x\to3^+} f(x) = f(3)$

$3m = 3 = n$

$\Rightarrow 3m = 3,\ n = 3$

$m = 1$

Q.4 Discuss the continuity of f(x) at x = 2 if

$$f(x) = \begin{cases} x^2 - 1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \geq 2 \end{cases}$$

(Multan Board 2012 G-I)(Bahawalpur Board 2019)

Sol: At x = 2

(i) f(2) = 3

(ii) $\lim_{x\to2^-} f(x) = \lim_{x\to2} (x^2 - 1) = (2)^2 - 1$

$= 4 - 1 = 3$

$\lim_{x\to2^+} f(x) = \lim_{x\to2} (3) = 3$

Since $\lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x)$

$\therefore \lim_{x\to2} f(x) = 3$

(iii) $\lim_{x\to2} f(x) = f(2)$

Hence f(x) is continuous at x = 2.

Now at x = −2

(i) f(−2) = 3(−2) = −6

(ii) $\lim_{x\to-2^-} f(x) = \lim_{x\to-2} (3x) = 3(-2) = -6$

$\lim_{x\to-2^+} f(x) = \lim_{x\to-2} (x^2 - 1)$

$= (-2)^2 - 1 = 4 - 1 = 3$

Since $\lim_{x\to-2^-} f(x) \neq \lim_{x\to-2^+} f(x)$

$\therefore \lim_{x\to-2} f(x)$ does not exist.

Hence f(x) is discontinuous at x = −2.

Q.5 Discuss the continuity at x = 2 if

$$f(x) = \begin{cases} 2x + 5 & \text{if } x \leq 2 \\ 4x + 1 & \text{if } x > 2 \end{cases}$$

(Sargodha Board 2013)(Bahawalpur Board 2016)

Sol: (i) f(2) = 2(2) + 5 = 4 + 5 = 9

(ii) $\lim_{x\to2^-} f(x) = \lim_{x\to2} (2x + 5) = 2(2) + 5$

$= 4 + 5 = 9$

$\lim_{x\to2^+} f(x) = \lim_{x\to2} (4x + 1) = 4(2) + 1$

$= 8 + 1 = 9$

Since $\lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x)$

$\therefore \lim_{x\to2} f(x) = 9$

(iii) $\lim_{x\to2} f(x) = f(2)$

Hence f(x) is continuous at x = 2.

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 The function $f(x) = \frac{x^2 - 1}{x - 1}$ is discontinuous at: *(Gujranwala Board 2013-16)*

(a) 0 (b) 1

(c) −1 (d) 2

Q.2 If f(x) is continuous at x = C, then: *(Lahore Board 2013 G-II)*

(a) $\lim_{x\to0} f(x) \neq 0$ (b) $\lim_{x\to C} f(x) = 1$

(c) $\lim_{x\to C} f(x) = f(C)$ (d) $\lim_{x\to C} f(x) \neq f(C)$

Q.3 The function $f(x) = \frac{2 + 3x}{2x}$ is not continuous at: *(Rawalpindi Board 2016)*

(a) x = −3 (b) x = −2/3

(c) x = 0 (d) x = 1

Q.4 $\lim_{n\to\infty} \left(\frac{5n + 1}{5n}\right)^n =$ *(Lahore Board 2017 G-II)*

(a) $e^{\frac{1}{5}}$ (b) e^5

(c) e^{-5} (d) $e^{-\frac{1}{5}}$

Q.5 If f(x) is continuous at point x = a, then *(Rawalpindi Board 2017 G-I)*

(a) $f(a) = \lim_{x\to a} f(x)$ (b) $f(a) = \lim_{x\to0} f(x)$

(c) $f(0) = \lim_{x\to0} f(x)$ (d) $f(x) = \lim_{x\to a} f(x)$

Q.6 $f(x) = \frac{x^2 - 9}{x - 3}$, $x \neq 3$, is discontinuous at x = 3, because: *(Lahore Board 2012 G-I)*

(a) $\lim_{x\to3} f(x) = f(3)$

(b) f(3) does not exist

(c) $\lim_{x\to C} f(x)$ does not exist

(d) None of these

FORMULA FOR DIFFERENTIATION

1. $\frac{d}{dx} c = 0.$
2. $\frac{d}{dx} x^n = n\, x^{n-1}$ (power rule).
3. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$. (sum rule)
4. $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x).$

(Difference rule)

5. $\frac{d}{dx}(c\, f(x)) = c \frac{d}{dx} f(x).$
6. $\frac{d}{dx}(f(x)\, g(x)) = f(x) \frac{d}{dx} g(x) + \frac{df(x)}{dx} . g(x)$

(Product rule)

7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{[g(x)]^2}$ (Quotient)
8. $\frac{d}{dx}(f \circ g\,(x)) = \frac{df}{dg} \frac{dg}{dx}$ (chain rule)
9. $\frac{d}{dx}\ [f(x)]^n = n[f(x)]^{n-1}\, f'(x)$
10. $\frac{d}{dx} \sin x = \cos x$
11. $\frac{d}{dx}\ \cos x = -\sin x$
12. $\frac{d}{dx} \tan x = \sec^2 x$
13. $\frac{d}{dx} \cot x = -\text{cosec}^2 x$
14. $\frac{d}{dx} \sec x = \sec x \tan x$
15. $\frac{d}{dx} \text{cosec } x = -\text{cosec } x \cot x$
16. $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$
17. $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}}$
18. $\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$
19. $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1 + x^2}$
20. $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}$
21. $\frac{d}{dx} \text{cosec}^{-1} x = \frac{-1}{x\sqrt{x^2 - 1}}$
22. $\frac{d}{dx} a^x = a^x \ln a$
23. $\frac{d}{dx} e^x = e^x$
24. $\frac{d}{dx} \log_e x = \frac{d}{dx} \ln x = \frac{1}{x}$
25. $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$
26. $\frac{d}{dx} \sin hx = \cos hx$
27. $\frac{d}{dx} \cos hx = \sin hx$
28. $\frac{d}{dx} \tan hx = \sec h^2 x$
29. $\frac{d}{dx} \cot hx = -\text{cosec } h^2 x$
30. $\frac{d}{dx} \sec hx = -\sec hx \tan hx$
31. $\frac{d}{dx} \text{cosec } hx = -\text{cosec } hx \cot hx$
32. $\frac{d}{dx} \sin h^{-1} x = \frac{1}{\sqrt{1 + x^2}}$
33. $\frac{d}{dx} \cos h^{-1} x = \frac{1}{\sqrt{x^2 - 1}}$
34. $\frac{d}{dx} \tan h^{-1} x = \frac{1}{1 - x^2}$, $|x| < 1$
35. $\frac{d}{dx} \cot h^{-1} x = \frac{1}{1 - x^2}$, $|x| > 1$
36. $\frac{d}{dx} \sec h^{-1} x = -\frac{1}{x\sqrt{1 - x^2}}$, $0 < x < 1$
37. $\frac{d}{dx} \text{cosec } h^{-1} x = -\frac{1}{|x|\sqrt{x^2 + 1}}$, $x \in R - \{0\}$

Chapter 2

DIFFERENTIATION

EXERCISE 2.1

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find by definition, the derivatives w .r .t. 'x' of x^m, $m \in N$ *(Lahore Board 2012 G-II)*

Sol: Let $y = x^m$

Taking increments on both sides, we have

$$y + \delta y = (x + \delta x)^m$$

$$\delta y = (x + \delta x)^m - y = (x + \delta x)^m - x^m$$

$$= x^m \left(1 + \frac{\delta x}{x}\right)^m - x^m = x^m \left[\left(1 + \frac{\delta x}{x}\right)^m - 1\right]$$

$$= x^m \left[1 + m\left(\frac{\delta x}{x}\right) + \frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots + \left(\frac{\delta x}{x}\right)^m - 1\right]$$

$$= x^m \left[m\left(\frac{\delta x}{x}\right) + \frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots + \left(\frac{\delta x}{x}\right)^m\right]$$

$$\delta y = x^m \left(\frac{\delta x}{x}\right)\left[m + \frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right) + \cdots + \left(\frac{\delta x}{x}\right)^{m-1}\right]$$

Dividing both sides by δx, we have

$$\frac{\delta y}{\delta x} = x^{m-1}\left[m + \frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right) + \cdots + \left(\frac{\delta x}{x}\right)^{m-1}\right]$$

Taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^{m-1}\left[m + \frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right) + \cdots + \left(\frac{\delta x}{x}\right)^{m-1}\right]$$

$$\frac{dy}{dx} = x^{m-1}[m + 0 + 0 + \cdots + 0] = m x^{m-1}$$

Q.2 Find by definition, the derivatives w .r .t. 'x' of x^{-100} *(Gujranwala Board 2012)*

Sol: Let $y = x^{-100}$

Taking increments on both sides, we have

$$y + \delta y = (x + \delta x)^{-100}$$

$$\delta y = (x + \delta x)^{-100} - y = (x + \delta x)^{-100} - x^{-100}$$

$$= x^{-100}\left(1 + \frac{\delta x}{x}\right)^{-100} - x^{-100}$$

$$= x^{-100}\left[\left(1 + \frac{\delta x}{x}\right)^{-100} - 1\right]$$

$$= x^{-100}\left[1 + (-100)\left(\frac{\delta x}{x}\right) + \frac{(-100)(-100-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots - 1\right]$$

$$= x^{-100}\left[(-100)\left(\frac{\delta x}{x}\right) + \frac{(-100)(-100-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \cdots\right]$$

$$= x^{-100}\left(\frac{\delta x}{x}\right)\left[(-100) + \frac{(-100)(-101)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\delta y = x^{-1-99}(\delta x)\left[(-100) + \frac{(-100)(-101)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

Dividing both sides by δx, we have

$$\frac{\delta y}{\delta x} = x^{-101}\left[(-100) + \frac{(-100)(-101)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} x^{-101}\left[(-100) + \frac{(-100)(-101)}{2!}\left(\frac{\delta x}{x}\right) + \cdots\right]$$

$$\frac{dy}{dx} = x^{-101}[(-100) + 0 + \cdots]$$

$$= -100x^{-101} = \frac{-100}{x^{101}}$$

Q.3 Find derivative of $\sqrt{x}$ at $x = a$ by first principle. *(Lahore Board 2011 G-II)*

Sol: $f(x) = \sqrt{x}$

$$f(x + \delta x) = \sqrt{x + \delta x}$$

$$f(x + \delta x) - f(x) = \sqrt{x + \delta x} - \sqrt{x}$$

$$= \frac{\left(\sqrt{x+\delta x} - \sqrt{x}\right)\left(\sqrt{x+\delta x} + \sqrt{\delta x}\right)}{\sqrt{x+\delta x} + \sqrt{\delta x}}$$

(Rationalizing the numerator)

$$= \frac{x + \delta x - x}{\sqrt{x+\delta x} + \sqrt{x}}$$

i.e., $$f(x + \delta x) - f(x) = \frac{\delta x}{\sqrt{x+\delta x} + \sqrt{x}} \quad \text{.....(1)}$$

Dividing both sides by δx, we have

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{\delta x}{\delta x\left(\sqrt{x+\delta x} + \sqrt{x}\right)}$$

$$= \frac{1}{\sqrt{x+\delta x} + \sqrt{x}}$$

Taking limit of both sides as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

i.e., $f'(x) = \frac{1}{2\sqrt{x}}$ and $f'(a) = \frac{1}{2\sqrt{a}}$

Q.4 Find $\frac{dy}{dx}$ by first principle $\sqrt{x+2}$

(Rawalpindi Board 2016)(Sahiwal Board 2018)

Sol: $y = \sqrt{x+2}$ .....(i)

Taking increments on both sides, we have

$y + \delta y = \sqrt{x+\delta x+2}$ .....(ii)

Subtracting (i) from (ii), we have

$$y + \delta y - y = \sqrt{x+\delta x+2} - \sqrt{x+2}$$

$$\delta y = \left(\sqrt{x+\delta x+2} - \sqrt{x+2}\right) \times \frac{\sqrt{x+\delta x+2}+\sqrt{x+2}}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

$$\delta y = \frac{(x+\delta x+2)-(x+2)}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

$$\delta y = \frac{x+\delta x+2-x-2}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

$$\delta y = \frac{\delta x}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

Dividing both sides by δx, we have

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x+\delta x+2}+\sqrt{x+2}}$$

Taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{1}{\delta x}\left\{\frac{1}{\sqrt{x+\delta x+2}+\sqrt{x+2}}\right\}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left\{\frac{1}{\sqrt{x+\delta x+2}+\sqrt{x+2}}\right\}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x+0+2}+\sqrt{x+2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x+2}+\sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}$$

Q.5 Find $\frac{dy}{dx}$ from first principles if $y = \sqrt{x+a}$

(Lahore Board 2010 G-I)

Sol: $y = \sqrt{x+a}$ .....(i)

Taking increments on both sides, we have

$y + \delta y = \sqrt{x+\delta x+a}$ .....(ii)

Subtracting (i) from (ii), we have

$$y + \delta y - y = \frac{1}{\sqrt{x+a+\delta x}} - \frac{1}{\sqrt{x+a}}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+a+\delta x}}{\left(\sqrt{x+a+\delta x}\right)\left(\sqrt{x+a}\right)}$$

$$\delta y = \frac{\left(\sqrt{x+a}-\sqrt{x+a+\delta x}\right)}{\left(\sqrt{x+a+\delta x}\right)\left(\sqrt{x+a}\right)} \times \frac{\sqrt{x+a}+\sqrt{x+a+\delta x}}{\sqrt{x+a}+\sqrt{x+a+\delta x}}$$

$$= \frac{\left(\sqrt{x+a}\right)^2 - \left(\sqrt{x+a+\delta x}\right)^2}{\left(\sqrt{x+a+\delta x}\right)\left(\sqrt{x+a}\right)\left(\sqrt{x+a}+\sqrt{x+a+\delta x}\right)}$$

$$= \frac{(x+a)-(x+a+\delta x)}{\left(\sqrt{x+a+\delta x}\right)\left(\sqrt{x+a}\right)\left(\sqrt{x+a}+\sqrt{x+a+\delta x}\right)}$$

$$= \frac{x+a-x-a-\delta x}{\left(\sqrt{x+a+\delta x}\right)\left(\sqrt{x+a}\right)\left(\sqrt{x+a}+\sqrt{x+a+\delta x}\right)}$$

$$= \frac{-\delta x}{\left(\sqrt{x+a+\delta x}\right)\left(\sqrt{x+a}\right)\left(\sqrt{x+a}+\sqrt{x+a+\delta x}\right)}$$

Dividing both sides by δx, we have

$$\frac{\delta y}{\delta x} = \frac{-1}{\left(\sqrt{x+a+\delta x}\right)\left(\sqrt{x+a}\right)\left(\sqrt{x+a}+\sqrt{x+a+\delta x}\right)}$$

Taking limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{-1}{\left(\sqrt{x+a+\delta x}\right)\left(\sqrt{x+a}\right)\left(\sqrt{x+a}+\sqrt{x+a+\delta x}\right)}$$

$$\frac{dy}{dx} = \frac{-1}{\left(\sqrt{x+a+0}\right)\left(\sqrt{x+a}\right)\left(\sqrt{x+a}+\sqrt{x+a+0}\right)}$$

$$= \frac{-1}{\left(\sqrt{x+a}\right)\left(\sqrt{x+a}\right)\left(\sqrt{x+a}+\sqrt{x+a}\right)}$$

$$= \frac{-1}{\left(\sqrt{x+a}\right)^2\left(2\sqrt{x+a}\right)} = \frac{-1}{(x+a)\left(2\sqrt{x+a}\right)}$$

$$= -\frac{1}{2(x+a)3/2}$$

Q.6 Find by definition, the derivative of $2 - \sqrt{x}$ w.r.t x.

Sol: Let $f(x) = 2 - \sqrt{x} = 2 - x^{1/2}$

$f(x+\delta x) = 2 - (x+\delta x)^{1/2}$

$f(x+\delta x) - f(x) = 2 - (x+\delta x)^{1/2} - 2 + x^{1/2}$

$= -(x+\delta x)^{1/2} + x^{1/2}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-(x+\delta x)^{1/2} + x^{1/2}}{\delta x}$$

$$= \frac{-x^{1/2}\left(1+\frac{\delta x}{x}\right)^{1/2} + x^{1/2}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-x^{1/2}\left[\left(1+\frac{\delta x}{x}\right)^{1/2} - 1\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-x^{1/2}\left[1 + \frac{1}{2}\left(\frac{\delta x}{x}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots\ldots - 1\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-x^{1/2}\left[\frac{1}{2}\left(\frac{\delta x}{x}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots\ldots\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-x^{1/2}\,\frac{\delta x}{x}\left[\frac{1}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{\delta x}{x} + \ldots\ldots\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-x^{-\frac{1}{2}}\,\delta x\left[\frac{1}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{\delta x}{x} + \ldots\ldots\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} -x^{-1/2}\left[\frac{1}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{\delta x}{x} + \ldots\ldots\right]$$

$$= -x^{-1/2}\cdot\left[\frac{1}{2} + 0 + 0 + \ldots\ldots\right]$$

$$= -x^{-1/2}\left[\frac{1}{2}\right] = -\frac{1}{2x^{1/2}} = -\frac{1}{2\sqrt{x}}$$

Q.7 Find by definition derivative of $\frac{1}{x-a}$.

(Faisalabad Board 2019)

Sol: Let

$$f(x) = \frac{1}{x-a} \quad \Rightarrow \quad f(x+\delta x) = \frac{1}{(x+\delta x-a)}$$

$$f(x+\delta x) - f(x) = \frac{1}{(x+\delta x-a)} - \frac{1}{(x-a)}$$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\frac{1}{x+\delta x-a} - \frac{1}{x-a}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\frac{x-a-x-\delta x+a}{(x-a)(x+\delta x-a)}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\frac{-\delta x}{(x-a)(x+\delta x-a)}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-\delta x}{\delta x(x+\delta x-a)(x-a)}$$

$$= \lim_{\delta x \to 0} \frac{-1}{(x+\delta x-a)(x-a)}$$

$$= \frac{-1}{(x+0-a)(x-a)} = \frac{-1}{(x-a)(x-a)}$$

$$= \frac{-1}{(x-a)^2}$$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 The notation used by Lagrange for derivative is:

(D.G.K Board 2014 G-II)

(a) $\frac{df}{dx}$ (b) f°(x)

(c) $f'(x)$ (d) Df(x)

Q.2 If $y = \sqrt{x}$, then $\frac{dy}{dx} =$ _____

(Multan Board 2013 G-II, D.G.K Board 2014 G-II)

(a) $2\sqrt{x}$ (b) $\frac{1}{2\sqrt{x}}$

(c) $\frac{2}{\sqrt{x}}$ (d) $\frac{2}{2\sqrt{x}}$

Q.3 $\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{x} =$ _____

(Multan Board 2014 G-II)

(a) $f'(x)$ (b) $f'(a)$

(c) $f'(2)$ (d) $f'(0)$

Q.4 If $f(x) = x^{100}$, then $f'(1) =$ _____

(Multan Board 2014 G-II)

(a) 100 (b) 99

(c) 50 (d) 0

Q.5 If $f(x) = x^{2/3}$, then $f'(8) =$ _____

(Lahore Board 2014 G-I, Bahawalpur Board 2014) (Sahiwal Board 2017)

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$

(c) $\frac{1}{3}$ (d) 3

Q.6 $\frac{d}{dx}x^3$ is equal to: *(Rawalpindi Board 2014)*

(a) $\frac{x^3}{3}$ (b) x^2

(c) $3x^2$ (d) $4x^4$

Q.7 Instantaneous rate of change of y with respect to x is given by: *(D.G.K Board 2015 G-II)*

(a) $\frac{\delta_y}{\delta_x}$ (b) $\frac{\delta_x}{\delta_y}$

(c) $\frac{dy}{dx}$ (d) $\frac{dx}{dy}$

Q.8 Newton used ——— for derivative of y: *(D.G.K Board 2012 G-I)*

(a) $\frac{dy}{dx}$ (b) y′

(c) y° (d) Dy

Q.9 $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ equals to: *(D.G.K Board 2010) (Faisalabad Board 2017)(Sahiwal Board 2017)*

(a) f′(0) (b) f′(a)

(c) f′(x) (d) f(x)

Q.10 $\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$ equals to: *(D.G.K Board 2010)*

(a) f(x) (b) f′(x)

(c) f′(a) (d) f′(0)

Q.11 Sir Issac Newton was a (an) ___ mathematician.

(a) German (b) English

(c) Swiss (d) French

Q.12 Got fried Wilhelm Leibniz was a (an) _____ mathematician.

(a) German (b) English

(c) Swiss (d) French

Q.13 The quantity $\frac{dy}{dx}$ is defined.

(a) $\lim_{\delta x\to 0}\frac{\delta x}{\delta y}$ (b) $\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$

(c) $\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$ (d) $\lim_{\delta x\to 0}\frac{\delta x}{\delta y}$

Q.14 If y = f(x), then $\frac{dy}{dx}$ is defined as.

(a) $\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{f(x-\delta x)-f(x)}{\delta x}$

(b) $\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{f(x+\delta x)+f(x)}{\delta x}$

(c) $\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{f(+\delta x)-f(x)}{\delta x}$

(d) $\lim_{\delta x\to 0}\frac{\delta x}{\delta y}=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta y}$

Q.15 If $\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$ exists, then it is defined to be the ——— of f with respect to x at x:

(a) Derivative (b) Increment

(c) Increasing (d) None of these

Q.16 $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ is called the derivative of f at ———

(a) at x = a (b) For all x

(c) at x → a (d) None of these

Q.17 ——— used $\frac{dy}{dx}$ symbol for the derivative of y = f(x) with respect to x:

(a) Lagrange (b) Newton

(c) Leibniz (d) Cauchy

Q.18 ——— used f(x) symbol for the derivative of y = f(x) with respect to x:

(a) Leibniz (b) Cauchy

(c) Lagrange (d) Newton

Q.19 ——— considered the problem of instantaneous rates of change:

(a) Sir Isaac Newton

(b) Gottfried Whilhelm Leibniz

(c) Sir Isaac Newton and Gottfried Whilhelm Leibniz

(d) None of the above

Q.20 ——— used the symbol Df(x) for the derivative of y = f(x) with respect to x:

(a) Newton (b) Leibniz

(c) Lagrange (d) Cauchy

Q.21 Which one is Leibniz notation derivative of function: *(Gujranwala Board 2014)*

(a) f °(x) (b) f ′(x)

(c) $\frac{d}{dx}f$ (d) Df(x)

Q.22 Notation for derivative was used by Newton is: *(Lahore Board 2013 G-I)*

(a) $\frac{dy}{dx}$ (b) Df(x)

(c) f°(x) (d) f′(x)

Q.23 Notation D f(x) for derivative was used by: *(Lahore Board 2012 G-II)*

(a) Cauchy (b) Newton

(c) Leibniz (d) Lagrange

Q.24 Which one is leibniz Notation for Derivative of f(fx): *(Bahawalpur Board 2019)*

(a) $\frac{df}{dx}$ (b) f(x)

(c) $\frac{d}{dx}$ (d) Df(x)

Q.25 The notation used for derivative of f(x) by Cauchy is: *(Faisalabad Board 2019 G-II)*

(a) Df (x) (b) f '(x)

(c) f(x) (d) $\frac{df}{dx}$

EXERCISE 2.2

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 $\frac{d}{dx}\left(\frac{1}{ax+b}\right)$ is equal to:

(Rawalpindi Board 2013, 2018)

(a) $\frac{1}{(ax+b)^2}$ (b) $\frac{a}{(ax+b)^2}$

(c) $\frac{-a}{(ax+b)^2}$ (d) $ln\,(ax+b)$

Q.2 $\frac{d}{dx}(cx+d)^{-n} = -nc\,(cx+d)^{-n-1}$, where m is any rational number is called ——— rule:

(a) Chain rule (b) Power rule

(c) Quotient rule (d) Product rule

EXERCISE 2.3

SHORT ANSWERS TO THE QUESTIONS

Q.1 Differentiate w. r. t. 'x' $\frac{2x-3}{2x+1}$

(Faisalabad Board 2017)(Sahiwal Board 2019) (D.G Khan Board 2014 G-II, 2017 G-I,II) (Lahore Board 2013 G-I(Faisalabad Board 2017)) (Gujranwala Board 2014)(Sargodha Board 2018) (Faisalabad Board 2019 G-II)

Sol: Let $y = \frac{2x-3}{2x+1}$

$$\frac{dy}{dx} = \frac{(2x+1)(2-0)-(2x-3)(2+0)}{(2x+1)^2}$$

$$= \frac{4x+2-4x+6}{(2x+1)^2} = \frac{8}{(2x+1)^2}$$

Q.2 Differentiate w. r. t. 'x' $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$

(Rawalpindi Board 2014)(Gujranwala Board 2010) (Lahore Board 2016, 2018 G-I) (Multan Board 2013, 2017, 2018 G-I)

Sol: Let $y = \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$

$$= \left(\sqrt{x}\right)^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2\left(\sqrt{x}\right)\left(\frac{1}{\sqrt{x}}\right) = x + \frac{1}{x} - 2$$

$$y = x + x^{-1} - 2$$

Differentiating w.r.t. 'x', we have

$$\frac{dy}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1}) - \frac{d}{dx}(2)$$

$$= 1 + (-1)x^{-1-1} - 0 = 1 - x^{-2} = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

Q.3 Differentiate $\frac{x^2+1}{x^2-3}$ w.r.t 'x'.

(Multan Board 2013)(Sahiwal Board 2013, 2018) (Lahore Board 2013, 2015 G-II)(Sargodha Board 2019)

Ans. Let $y = \frac{x^2+1}{x^2-3}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-3}\right)$$

$$= \frac{(x^2-3)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-3)}{(x^2-3)^2}$$

$$= \frac{(x^2-3)(2x+0) - (x^2+1)(2x-0)}{(x^2-3)^2}$$

$$= \frac{2x^3-6x-2x^3-2x}{(x^2-3)^2} = \frac{8x}{(x^2-3)^2}$$

Q.4 Find derivative of $\sqrt{\frac{a-x}{a+x}}$

(Lahore Board 2017 G-I)(Lahore Board 2019 G-II)

Sol: Let $y = \sqrt{\frac{a-x}{a+x}}$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}-1} \frac{d}{dx}\frac{a-x}{a+x}$$

$$= \frac{1}{2}\left(\frac{a-x}{a+x}\right)^{\frac{-1}{2}} \frac{(a+x)\frac{d}{dx}(a-x) - (a-x)\frac{d}{dx}(a+x)}{(a+x)^2}$$

$$= \frac{1}{2}\frac{(a-x)^{\frac{-1}{2}}}{(a+x)^{\frac{-1}{2}}} \cdot \frac{(a+x)(0-1)-(a-x)(0+1)}{(a+x)^2}$$

$$= \frac{1}{2} \cdot \frac{1}{(a-x)^{\frac{1}{2}}} \times \frac{-a-x-a+x}{(a+x)^{2-\frac{1}{2}}}$$

$$= \frac{1}{2} \times \frac{1}{(a-x)^{\frac{1}{2}}} \times \frac{-2a}{(a+x)^{\frac{3}{2}}} = \frac{-a}{(a-x)^{\frac{1}{2}}(a+x)^{\frac{3}{2}}}$$

Q.5 Differentiate $\frac{(x^2+1)^2}{x^2-1}$ w.r.t. 'x'

(Gujranwala Board 2016)(Faisalabad Board 2019)

Sol: Let $y = \frac{(x^2+1)^2}{x^2-1}$

$$\frac{dy}{dx} = \frac{(x^2-1)\,2(x^2+1)\frac{d}{dx}(x^2+1) - (x^2+1)^2(2x)}{(x^2-1)^2}$$

$$= \frac{2(x^2-1)(x^2+1)(2x)-(x^2+1)^2(2x)}{(x^2-1)^2}$$

$$= \frac{2x(x^2+1)[2(x^2-1)-(x^2+1)]}{(x^2-1)^2}$$

$$= \frac{2x(x^2+1)[2x^2-2-x^2-1]}{(x^2-1)^2} = \frac{2x(x^2+1)(x^2-3)}{(x^2-1)^2}$$

Q.6 Differentiate $\frac{2x^3-3x^2+5}{x^2+1}$ with respect to x.

(Multan Board 2014 G-I)

Sol: Let $f(x) = \frac{2x^3-3x^2+5}{x^2+1}$

We take $f(x) = 2x^3-3x^2+5$ and $g(x) = x^2+1$

Then $f'(x) = \frac{d}{dx}[2x^3-3x^2+5]$

$= 2(3x^2)-3(2x)+0 = 6x^2-6x$

and $g'(x) = \frac{d}{dx}[x^2+1] = 2x+0 = 2x$

Using the formula

$f'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$, we obtain

$$\frac{d}{dx}\left[\frac{2x^3-3x^2+5}{x^2+1}\right] = \frac{(6x^2-6x)(x^2+1)-(2x^3-3x^2+5)(2x)}{(x^2+1)^2}$$

$$= \frac{6x^4-6x^3+6x^2-6x-(4x^4-6x^3+10x)}{(x^2+1)2}$$

$$= \frac{6x^4-6x^3+6x^2-6x-4x^4+6x^3-10x}{(x^2+1)2} = \frac{2x^4+6x^2-16x}{(x^2+1)^2}$$

Q.7 If $y = x^4+2x^2+2$, prove that

$\frac{dy}{dx} = 4x\sqrt{y-1}$ ***(Lahore Board 2014 G-I)***

(Bahawalpur Board 2019)(Faisalabad Board 2019 G-II)

Sol: $y = x^4+2x^2+2$

Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4)+2\frac{d}{dx}(x^2)+\frac{d}{dx}(2)$$

$$= 4x^3+4x+0 = 4x(x^2+1) = 4x\sqrt{(x^2+1)^2}$$

$$= 4x\sqrt{x^4+2x^2+1} = 4x\sqrt{(x^4+2x^2+2)-1}$$

$$\frac{dy}{dx} = 4x\sqrt{y-1} \quad (\because y = x^4+2x^2+2)$$

Q.8 If $y = \sqrt{x}-\frac{1}{\sqrt{x}}$, show that $2x\frac{dy}{dx}+y = 2\sqrt{x}$

(Multan Board 2014, 2018 G-II)

Sol: $y = \sqrt{x}-\frac{1}{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x})-\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right)-\frac{d}{dx}\left(x^{-\frac{1}{2}}\right)$$

$$= \frac{1}{2}x^{\frac{1}{2}-1}-\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}$$

$$= \frac{1}{2}\left(x^{-\frac{1}{2}}+x^{-\frac{3}{2}}\right)$$

Multiplying both sides by 2, we have

$$2\frac{dy}{dx} = x^{-\frac{1}{2}}+x^{-\frac{3}{2}}$$

$$2\frac{dy}{dx} = \frac{1}{\sqrt{x}}+\frac{1}{x\sqrt{x}}$$

Multiplying both sides by x, we have

$$2x\frac{dy}{dx} = \frac{x}{\sqrt{x}}+\frac{x}{x\sqrt{x}}$$

$$2x\frac{dy}{dx} = \sqrt{x}+\frac{1}{\sqrt{x}}$$

Adding y on both sides, we have

$$2x\frac{dy}{dx}+y = \sqrt{x}+\frac{1}{\sqrt{x}}+y$$

$$2x\frac{dy}{dx}+y = \sqrt{x}+\frac{1}{\sqrt{x}}+\sqrt{x}-\frac{1}{\sqrt{x}} \quad \left(\Theta\ y = \sqrt{x}-\frac{1}{\sqrt{x}}\right)$$

$$2x\frac{dy}{dx}+y = 2\sqrt{x}$$

LONG QUESTIONS

Q.1 Differentiate $\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$ w.r.t x.

(Gujranwala Board 2018)

Sol: Let $y = \frac{\sqrt{x^2+1}}{\sqrt{x^2-1}} = \sqrt{\frac{x^2+1}{x^2-1}} = \left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{2}}$

Differentiating w.r.t "x", we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{\frac{-1}{2}}\frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2+1)^2}$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{\frac{-1}{2}}\frac{(x^2-1)2x-(x^2+1)2x}{(x^2+1)^2}$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{\frac{-1}{2}}\frac{2x(x^2-1-x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2+1)^{\frac{-1}{2}}}{(x^2-1)^{\frac{-1}{2}}}\frac{x(-2)}{(x^2-1)^2} = \frac{-2x(x^2+1)^{\frac{-1}{2}}}{(x^2-1)^{\frac{1}{2}}(x^2-1)^2}$$

$$= \frac{-2x}{(x^2+1)^{\frac{1}{2}}(x^2-1)^{\frac{-1}{2}}} = \frac{-2x}{\sqrt{x^2+1}(x^2-1)^{\frac{3}{2}}}$$

Q.2 If $y = x^4+2x^2+2$, then prove that

$\frac{dy}{dx} = 4x\sqrt{y-1}$ ***(Multan Board 2015 G-II)***

Sol: See Short Question 18

Q.3 Differentiate $\frac{(\sqrt{x}+1)(x^{3/2}-1)}{x^{3/2}-x^{1/2}}$ with respect to x.

(Multan Board 2012 G-II)

Sol: Let $y = \frac{(\sqrt{x}+1)(x^{3/2}-1)}{(x^{3/2}-x^{1/2})}$

$= \frac{(x^{1/2}+1)((x^{1/2})^3-(1)^3)}{x^{1/2}(x-1)}$

$= \frac{(x^{1/2}+1)(x^{1/2}-1)(x+x^{1/2}+1)}{x^{1/2}(x-1)}$

$= \frac{(x-1)(x+x^{1/2}+1)}{x^{1/2}(x-1)}$

$= \frac{x+x^{1/2}+1}{x^{1/2}} = \frac{x}{x^{1/2}}+\frac{x^{1/2}}{x^{1/2}}+\frac{1}{x^{1/2}}$

$= x^{1/2}+1+x^{-1/2}$

Differentiating w.r.t "x", we have

$\frac{dy}{dx} = \frac{d}{dx}[x^{1/2}+1+x^{-1/2}]$

$= \frac{d}{dx}(x^{1/2})+\frac{d}{dx}(1)+\frac{d}{dx}(x^{-1/2})$

$= \frac{1}{2}x^{-1/2}+0+\frac{-1}{2}x^{-3/2} = \frac{1}{2x^{1/2}}-\frac{1}{2x^{3/2}}$

$= \frac{1}{2x^{1/2}}-\frac{1}{2x\,x^{1/2}} = \frac{x-1}{2x\,x^{1/2}} = \frac{x-1}{2x^{3/2}}$

Q.4 Find $\frac{dy}{dx}$ if $y = \frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$, $x \neq 0$

(Gujranwala Board 2019 G-II)

Sol: Let $y = \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+(1-x)}$

$= \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \times \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$

$= \frac{(\sqrt{1+x}-\sqrt{1-x})^2}{(\sqrt{1+x})^2-(\sqrt{1-x})^2}$

$= \frac{(\sqrt{1+x})^2+(\sqrt{1-x})^2-2(\sqrt{1+x})(\sqrt{1-x})}{(1+x)-(1-x)}$

$= \frac{1+x+1-x-2\sqrt{1-x^2}}{1+x-1+x} = \frac{2-2\sqrt{1-x^2}}{2x}$

$= \frac{2(1-\sqrt{1-x^2})}{2x} = \frac{1-\sqrt{1-x^2}}{x} = \frac{1-\sqrt{1-x^2}}{x}$

Differentiating w.r.t "x", we have

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1-\sqrt{1-x^2}}{x}\right)$

$\frac{dy}{dx} = \frac{(x)\frac{d}{dx}(1-\sqrt{1-x^2})-(1-\sqrt{1-x^2})\frac{d}{dx}(x)}{x^2}$

$= \frac{(x)\left(-\frac{1}{2\sqrt{1-x^2}}(-2x)\right)-(1-\sqrt{1-x^2})(1)}{x^2}$

$= \frac{\frac{x^2}{\sqrt{(1-x^2)}}-1+\sqrt{1-x^2}}{x^2} = \frac{\frac{x^2-\sqrt{1-x^2}+1-x^2}{\sqrt{1-x^2}}}{x^2}$

$= \frac{1-\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 $\frac{1}{1+x^2}$ is derivative of:

(Multan Board 2014 G-II, D.G.K Board 2014 G-II)

(a) $\sin^{-1}x$ (b) $\sec^{-1}x$

(c) $\tan^{-1}x$ (d) $\cot^{-1}x$

Q.2 The derivative of $\frac{x^3+2x^2}{x^3}$ equals:

(Multan Board 2014 G-I)

(a) $\frac{2}{x^2}$ (b) $\frac{-2}{x^2}$

(c) $\frac{1}{2x^2}$ (d) $\frac{-1}{2x^2}$

Q.3 $\frac{d}{dx}\left(\frac{g(x)}{f(x)}\right)$ is equal: *(Rawalpindi Board 2014)*

(Sahiwal Board 2017)

(a) $\frac{f(x)\,g'(x)-g(x)\,f'(x)}{(f(x))^2}$

(b) $\frac{g(x)\,f'(x)-f(x)\,g'(x)}{(f(x))^2}$

(c) $\frac{g(x)\,f'(x)-f(x)\,g'(x)}{(f(x))^2}$

(d) $\frac{f(x)\,g'(x)-g(x)\,f'(x)}{(g(x))^2}$

Q.4 If $y = \sqrt{x}$, then $\frac{dy}{dx} =$ *(Multan Board 2013 G-I)*

(D.G. Khan Board 2015 G-I)(Lahore Board 2019)

(a) $\frac{1}{\sqrt{x}}$ (b) $\frac{1}{2\sqrt{x}}$

(c) $\frac{\sqrt{x}}{2}$ (d) $\frac{2}{\sqrt{x}}$

Q.5 Derivative of $\frac{x-1}{x+1}$ w.r.t x is:

(Sargodha Board 2013)

(a) $\frac{-1}{(x+1)^2}$ (b) $\frac{-2}{(x+1)^2}$

(c) $\frac{2}{(x+1)^2}$ (d) $\frac{-2x+1}{(x+1)^2}$

Q.6 The derivative of $\sqrt{x}$ at x = a:
(Faisalabad Board 2013)

(a) $\sqrt{2a}$ (b) $\frac{1}{\sqrt{2a}}$

(c) $\sqrt[2]{a}$ (d) $\frac{1}{2\sqrt{a}}$

Q.7 $\frac{d}{dx}\sqrt{x} =$
(Gujranwala Board 2018, D.G. Khan Board 2015 G-I)

(a) $\frac{1}{2}\sqrt{x}$ (b) $\frac{-1}{2}x$

(c) $\frac{1}{2}x^{-3/2}$ (d) $\frac{1}{2\sqrt{x}}$

Q.8 $\frac{d}{dx}(x-a)^{1/2}$ equals: *(D.G.K Board 2013 G-II)*

(a) $\frac{2}{\sqrt{x-a}}$ (b) $\frac{-1}{2\sqrt{x-a}}$

(c) $\frac{1}{2\sqrt{x-a}}$ (d) $\sqrt{x-a}$

Q.9 $\frac{d}{dx}(f(x) \cdot g(x)) = ?$ *(D.G.K Board 2013 G-II)*

(a) $f(x)\,g'(x) - f'(x)\,g(x)$
(b) $f'(x)\,g(x) - f(x)\,g'(x)$
(c) $f'(x)\,g(x) + f(x)\,g'(x)$
(d) $f'(x)\,g'(x)$

Q.10 $\frac{d}{dx}x^n = ?$ *(D.G.K Board 2013 G-I)*
(Lahore Board 2015 G-I)

(a) $-n\,x^{n-1}$ (b) $n\,x^{n-1}$

(c) $(n-1)x^{n-1}$ (d) $\frac{x^{n-1}}{n+1}$

Q.11 $\frac{d}{dx}(f(x))^n = ?$ *(D.G.K Board 2013 G-I)*

(a) $n(f(x))^{n-1}$ (b) $(f'(x))^n$

(c) $n(f(x))^{n-1}f'(x)$ (d) $\frac{f'(x)}{n}$

Q.12 $\frac{d}{dx}\{c.f(x)\}$:

(a) c (b) $\frac{d}{dx}f(x)$

(c) $c\left\{\frac{d}{dx}f(x)\right\}$ (d) f(x)

Q.13 $\{f(x) - g(x)\}' =$ *(Sargodha Board 2017)*

(a) $f'(x) - g'(x)$ (b) $f'(x) \times g'(x)$

(c) $f'(x) + g'(x)$ (d) $f'(x) \pm g'(x)$

Q.14 $[fog(x)]' =$

(a) $f'\{g(x)\}$ (b) $f'\{g'(x)\}$

(c) $f'\{g(x)\} \times g'(x)$ (d) None of these

Q.15 $\frac{d}{dx}\{f(x) + g(x)\} =$

(a) $\frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ (b) $\frac{d}{dx}f(x) \times \frac{d}{dx}g(x)$

(c) $\frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$ (d) $\frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

Q.16 $\frac{d}{dx}(ax+b)^n = na\,(ax+b)^{n-1}$, where n is any rational number is called:

(a) Power rule (b) Chain rule
(c) Power and chain rule
(d) None of these

Q.17 $\frac{d}{dx}(c) =$ _____, where c is any constant:

(a) c (b) 0
(c) 1 (d) –c

Q.18 $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \times \left\{\frac{d}{dx}g(x)\right\}$ where n is any rational number is called _____ rule:

(a) Power rule (b) Chain rule
(c) Power and Chain rule
(d) None of these

Q.19 if x and y are the functions of u, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ is called _____ rule:

(a) Power rule (b) Derivative rule
(c) Chain rule (d) None of these

Q.20 If f(x) have second derivative at c such that $f'(c) = 0$ and $f''(c) < 0$, then c is a point of:

(a) Minima (b) Maxima
(c) Point of inflection (d) None of these

Q.21 Let y = f(x) be an increasing function of x. Then

(a) $\frac{dy}{dx} = 0$ (b) $\frac{dy}{dx} \le 0$

(c) $\frac{dy}{dx} > 0$ (d) None of these

Q.22 If $y = \frac{(x^2+1)(x+3)}{x}$, $\frac{dy}{dx}$ is equal to:

(a) $2x + 3 - \frac{1}{x^2}$ (b) $3x + 3 - \frac{3}{x^2}$

(c) $2x + 3 - \frac{3}{x^2}$ (d) None of these

Q.23 $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\left\{\frac{d}{dx}f(x)\right\} \times g(x) - f(x) \times \left\{\frac{d}{dx}g(x)\right\}}{[g(x)]^2}$, provided:

(a) $g(x) \neq f(x)$ (b) $g(x) \neq 0$
(c) $g(x) = f(x)$ (d) $g(x) = 0$

Q.24 If s is the distance traveled by a body at time t, the velocity is given by the expression:

(a) $\frac{\delta s}{\delta t}$ (b) $\frac{dt}{ds}$

(c) $\frac{ds}{dt}$ (d) $\frac{\delta t}{\delta s}$

Q.25 If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, then

(a) $2\sqrt{x}\frac{dy}{dx} + y = 2x$ (b) $2x\frac{dy}{dx} + y = 2\sqrt{x}$

(c) $2y\frac{dy}{dx} + x = \sqrt{x}$ (d) $2x\frac{dy}{dx} - y = 2\sqrt{x}$

Q.26 If $y = \sqrt{1-x^2}$, $0 < x < 1$, then $\frac{dy}{dx}$ is equal to:

(a) $\sqrt{x^2-1}$ (b) $\frac{1}{\sqrt{x^2-1}}$

(c) $\frac{x}{\sqrt{1-x^2}}$ (d) $-\frac{x}{\sqrt{1-x^2}}$

Q.27 If $y = (ax)^n + b^n$, then $\frac{dy}{dx}$ is given by:

(a) $n(ax)^{n-1}$ (b) $n\,x^n\,x^{n-1}$

(c) $n\,a^n\,x^{n-1}$ (d) $a^n\,x^{n-1} + b$

Q.28 $\frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 =$

(a) $1 - \frac{1}{x^2}$ (b) $1 + \frac{1}{x^2}$

(c) $1 - \frac{1}{2x}$ (d) None of these

Q.29 $\frac{d}{dx}(2x^2+3)^5 =$ ———

(Gujranwala Board 2014) (Bahawalpur Board 2019)

(a) $5(2x^2+3)^4$ (b) $5(2x^2+4)^4\,4$

(c) $5(2x^2+3)^4\,3$ (d) $(2x^2+3)^4\,20x$

Q.30 $\frac{d}{dx}\left(\frac{x^2-4}{x-2}\right)$ equals: *(Lahore Board 2014 G-I)*

(a) 0 (b) 1

(c) $x + 2$ (d) $x - 2$

Q.31 $\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} =$ *(Lahore Board 2013 G-II)*

(a) f(x) (b) f′(x)

(c) f″(x) (d) f‴(x)

Q.32 $\frac{d}{dx}[Cf(x)] =$ *(Lahore Board 2012 G-I)*

(a) $c\,f'(x)$ (b) $C'\,f'(x)$

(c) $C'\,f(x)$ (d) All of these

Q.33 If $f(x) = 3 - \sqrt{x}$, then $f'(1) =$

(Faisalabad Board 2016)

(a) $\frac{3}{2}$ (b) $\frac{1}{2}$

(c) $-\frac{1}{2}$ (d) $\frac{-1}{\sqrt{2}}$

Q.34 $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) =$ *(Sargodha Board 2016)*

(a) $\frac{1}{2x\sqrt{x}}$ (b) $\frac{-1}{2x\sqrt{x}}$

(c) $\frac{1}{2}x\sqrt{x}$ (d) None of these

Q.35 $\frac{d}{dx}\left(\frac{1}{g(x)}\right) =$ *(Faisalabad Board 2017)*

(a) $\frac{g'(x)}{g(x)}$ (b) $-\frac{g'(x)}{g(x)}$

(c) $-\frac{g'(x)}{(g(x))^2}$ (d) $\frac{g'(x)}{(g(x))^2}$

Q.36 $\frac{d}{dx}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$ *(Lahore Board 2017 G-I)*

(a) $1 - \frac{1}{2x}$ (b) $1 + \frac{1}{x^2}$

(c) 0 (d) $1 - \frac{1}{x^2}$

Q.37 $\frac{d}{dx}(3x^{\frac{4}{3}}) =$ *(Sargodha Board 2017)*

(a) $4x$ (b) $4x^{\frac{1}{3}}$

(c) $4x$ (d) $4x^{-\frac{1}{3}}$

Q.38 $\frac{d}{dx}\sqrt{x} = ?$ at $x = a$ *(D.G.K Board 2012 G-I)*

(Sahiwal Board 2017)

(a) $\frac{1}{2\sqrt{a}}$ (b) $2\sqrt{a}$

(c) $\sqrt{2a}$ (d) $\frac{1}{\sqrt{2a}}$

Q.39 $\frac{d}{dx}\left(x - \frac{1}{x}\right) =$ *(D.G.K Board 2014 G-I)*

(Gujranwala Board 2016)

(a) $1 - \frac{1}{x}$ (b) $1 + \frac{1}{x}$

(c) $1 + \frac{1}{x^2}$ (d) $1 - \frac{1}{x^2}$

Q.40 If $y = \frac{1}{x^2}$, then $\frac{dy}{dx}$ at $x = -1$ is. *(A.J.K Board 2017)*

(a) 2 (b) 3

(c) $\frac{1}{3}$ (d) 4

Q.41 $\frac{d}{dx}\left(\frac{1}{x^2}\right)$ at $x = 1$ is.

(a) -2 (b) 2

(c) 1 (d) -1

Q.42 If $y = \sqrt{1-x^2}$, $0 < x < 1$, then $\frac{dy}{dx} =$.

(Lahore Board 2018 G-I)

(a) $\sqrt{x^2-1}$ (b) $\frac{1}{\sqrt{1-x^2}}$

(c) $\frac{x}{\sqrt{1-x^2}}$ (d) $\frac{-x}{\sqrt{1-x^2}}$

Q.43 If $y = x + \frac{1}{x}$, then $\frac{dy}{dx}$: *(Sahiwal Board 2013)*

(a) $1 + \frac{1}{x^2}$ (b) $1 + \frac{1}{x}$

(c) $1 - \frac{1}{x^2}$ (d) $1 - \frac{1}{x^2}$

Q.44 If $f(x) = \frac{1}{x^2}$, then $f'(-1) =$ *(Sargodha Board 2013)*

(a) 1 (b) -1

(c) $\frac{1}{2}$ (d) 2

Q.45 If $y = x^{-3/2}$, then $\frac{dy}{dx}$ is

(Multan Board 2013 G-I)

(a) $\frac{-3}{2}x^{-1/2}$ (b) $\frac{-3}{2}x^{1/2}$

(c) $-3x^{5/2}$ (d) $\frac{-3}{2}x^{-5/2}$

Q.46 $\frac{d}{dx}\left(\frac{1}{x^2}\right)$ at $x = 1$ is

(Gujranwala Board 2019 G-I)

(a) -2 (b) 2

(c) 1 (d) -1

Q.47 If $y = \frac{1}{x^2}$ then $\frac{dy}{dx}$ at $x = -1$

(Gujranwala Board 2019 G-II)

(a) 3 (b) $\frac{1}{3}$

(c) 2 (d) $\frac{1}{2}$

Q.48 $\frac{d}{dx}(x-5)(3-x) =$ *(Rawalpindi Board 2019)*

(a) $2x + 8$ (b) $-2x + 8$

(c) $2x - 8$ (d) $x + 8$

Q.49 $\frac{d}{dx}\left(\frac{a}{x}\right) =$ *(Sahiwal Board 2019)*

(a) a (b) $\frac{1}{x}$

(c) $\frac{a}{x^2}$ (d) $-\frac{a}{x^2}$

EXERCISE 2.4

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find $\frac{dx}{dt}$ and $\frac{dy}{dx}$ when $x = at^2$, $y = 2at$.

(Lahore Board 2013)(Sahiwal Board 2018)

Ans. $x = at^2$ | $y = 2at$

Diff. w.r.t. 't' | Diff. w.r.t. 't'

$\frac{dx}{dt} = 2at$ | $\frac{dy}{dt} = 2a$

Q.2 Differentiate $x^2 - \frac{1}{x^2}$ w.r.t x^4

(Multan Board 2014 G-I, G-II)(Multan Board 2013) (Rawalpindi Board 2016)(Faisalabad Board 2019 G-II)

Sol: Let $y = x^2 - \frac{1}{x^2}$ ..... (i)

and $u = x^4$ ..... (ii) Thus we have to find $\frac{dy}{du}$

Differentiating (i) w.r.t. 'x', we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 - \frac{1}{x^2}\right) = \frac{d}{dx}\left(x^2 - x^{-2}\right)$$

$$= \frac{d}{dx}(x^2) - \frac{d}{dx}(x^{-2})$$

$$= 2x - (-2x^{-2-1}) = 2x + 2x^{-3} = 2x + \frac{2}{x^3}$$

$$= \frac{2x^4 + 2}{x^3} = \frac{2(x^4+1)}{x^3}$$

Differentiating (ii) w.r.t. 'x', we have $\frac{du}{dx} = 4x^3$

By chain rule, we have

$$\frac{dy}{du} = \frac{\frac{dy}{dx}}{\frac{du}{dx}} = \frac{\frac{2(x^4+1)}{x^3}}{4x^3} = \frac{2(x^4+1)}{x^3} \times \frac{1}{4x^3} = \frac{x^4+1}{2x^6}$$

Q.3 Differentiate $x^2-\frac{1}{x^2}$ w.r.t x^2

(Gujranwala Board 2016)

Sol: Let $y = x^2-\frac{1}{x^2}$ ..... (i)

and $u = x^2$ ..... (ii) Thus we have to find $\frac{dy}{du}$

Differentiating (i) w.r.t. 'x', we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2-\frac{1}{x^2}\right) = \frac{d}{dx}\left(x^2 - x^{-2}\right)$$

$$= \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}\left(x^{-2}\right)$$

$$= 2x - \left(-2x^{-2-1}\right) = 2x + 2x^{-3}$$

$$= 2x + \frac{2}{x^3} = \frac{2x^4+2}{x^3} = \frac{2(x^4+1)}{x^3}$$

Differentiating (ii) w.r.t. 'x', we have $\frac{du}{dx} = 2x$

By chain rule, we have

$$\frac{dy}{du} = \frac{\frac{dy}{dx}}{\frac{du}{dx}} = \frac{\frac{2(x^4+1)}{x^3}}{2x} = \frac{2\left(x^4+1\right)}{x^3}\times\frac{1}{2x} = \frac{x^4+1}{x^4}$$

Q.4 Find $\frac{dy}{dx}$ if $3x + 4y + 7 = 0$

(Lahore Board 2016 G-I)(Bahawalpur Board 2016) (Faisalabad Board 2017)(D.G Khan Board 2017) (Sahiwal Board 2018)

Sol: $\frac{d}{dx}(3x + 4y + 7) = \frac{d}{dx}(0)$

$$\frac{d}{dx}(3x) + \frac{d}{dx}(4y) + \frac{d}{dx}(7) = 0$$

$$3(1) + 4\frac{dy}{dx} + 0 = 0 \Rightarrow 4\frac{dy}{dx} = -3 \Rightarrow \frac{dy}{dx} = -\frac{3}{4}$$

Q.5 Find $\frac{dy}{dx}$ if $x^2 - 4xy - 5y = 0$

(Lahore Board 2013, 2019 G-II) (Gujranwala Board 2010)(Rawalpindi Board 2019 G-II)

Sol: Differentiating the both sides w.r.t 'x', we have

$$\frac{d}{dx}(x^2 - 4xy - 5y) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(4xy) - \frac{d}{dx}(5y) = 0$$

$$2x - 4\left[x\cdot\frac{dy}{dx} + y(1)\right] - 5\frac{dy}{dx} = 0$$

$$2x - 4x\frac{dy}{dx} - 4y - 5\frac{dy}{dx} = 0$$

$$4x\frac{dy}{dx} + 5\frac{dy}{dx} = 2x - 4y$$

$$\frac{dy}{dx}(4x + 5) = 2(x - 2y) \Rightarrow \frac{dy}{dx} = \frac{2(x-2y)}{4x+5}$$

Q.6 Find $\frac{dy}{dx}$, if $xy+y^2 = 2$. *(Rawalpindi Board 2013)*

(D.G.K Board 2014 G-II) (Sahiwal Board 2014) (Multan Board 2013 G-II)(Sargodha Board 2018, 2019) (D.G Khan Board 2017 G-II)(Rawalpindi Board 2017) (Gujranwala Board 2017, 2019)

Ans. $xy + y^2 = 2$

Diff. w.r.t 'x'.

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}2$$

$$\frac{d}{dx}(xy) + \frac{d}{dx}y^2 = 0$$

$$x\frac{dy}{dx} + y(1) + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x+2y} = \frac{-y}{x+2y}$$

Q.7 Find $\frac{dy}{dx}$, if $y^2 + x^2 - 4x = 5$.

(Sargodha Board 2013) (Lahore Board 2014 G-II) (Multan Board 2014, 2017 G-I, 2018 G-II)

Ans. $y^2 + x^2 - 4x = 5$

Diff. w.r.t 'x'.

$$\frac{d}{dx}(y^2 + x^2 - 4x) = \frac{d}{dx}(5)$$

$$2y\frac{dy}{dx} + 2x - 4 = 0$$

$$2y\frac{dy}{dx} = 4 - 2x$$

$$\frac{dy}{dx} = \frac{2(2-x)}{2y} = \frac{2-x}{y}$$

Q.8 Find $\frac{dy}{dx}$, if $x = 1 - t^2$, $y = 3t^2 - 2t^3$

(Rawalpindi Board 2013) (Lahore Board 2010 G-II) (Bahawalpur Board 2016, 2018)

Ans. $x = 1 - t^2$

Diff. w.r.t. 'x'

$$\frac{dx}{dt} = \frac{d}{dt}(1 - t^2)$$

$$\frac{dx}{dt} = -2t$$

$y = 3t^2 - 2t^3$

Diff. w.r.t 'x'

$$\frac{dy}{dt} = \frac{d}{dt}(dt^2 - 2t^3)$$

$$\frac{dy}{dt} = 3(2t) - 2(3t^2) = 6t - 6t^2 = 6t(1 - t)$$

By chain rule.

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{6(1-t)t}{-2t} \Rightarrow \frac{dy}{dx} = -3(1 - t)$$

Q.9 Find $\frac{dy}{dx}$ when $y = (3x^2 - 2x + 7)^6$

(Bahawalpur Board 2016)

Sol: $y = (3x^2 - 2x + 7)^6$ Substituting

$u = 3x^2 - 2x + 7$, we have $y = u^6$

Differentiating $u = 3x^2 - 2x + 7$ w.r.t. 'x', we have

$$\frac{du}{dx} = 6x - 2$$

Differentiating $y = u^6$ w.r.t. 'u', we have $\frac{dy}{du} = 6u^5$

By chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 6u^5 \times (6x - 2) = 6(3x^2 - 2x + 7)^5 \times 2(3x - 1)$

$= 12(3x^2 - 2x + 7)^5 (3x - 1)$

Q.10 Differentiate x and y w.r.t t if

$$x = \frac{1 - t^2}{1 + t^2}, y = \frac{2t}{1 + t^2}$$

(Multan Board 2018 G-II)

Sol: See Long Question 5

LONG QUESTIONS

Q.1 Differentiate $x^2 - \frac{1}{x^2}$ w.r.t x^4.

(Multan Board 2014 G-II)

Sol: Let $y = x^2 - \frac{1}{x^2}$, $u = x^4$

$y = x^2 - x^{-2}$

Differentiating w.r.t "x", we have

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 - x^{-2})$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) - \frac{d}{dx}(x^{-2})$$

$$\frac{dy}{dx} = 2x - (-2x^{-3}) = 2x + \frac{2}{x^3} = 2\left(\frac{x^4 + 1}{x^3}\right)$$

$u = x^4$

Differentiating w.r.t "x", we have

$$\frac{du}{dx} = \frac{d}{dx}(x^4)$$

$$\frac{du}{dx} = 4x^3$$

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$ (By Chain rule)

$$\frac{dy}{du} = 2\left(\frac{x^4 + 1}{x^3}\right) \times \frac{1}{4x^3} = \frac{x^4 + 1}{2x^6}$$

Q.2 Differentiate $\frac{x^2 + 1}{x^2 - 1}$ w.r.t $\frac{x - 1}{x + 1}$.

(Lahore Board 2014 G-I)

Sol: Let $y = \frac{x^2 + 1}{x^2 - 1}$, $u = \frac{x - 1}{x + 1}$

$y = \frac{x^2 + 1}{x^2 - 1}$

Differentiating w.r.t "x", we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2 + 1}{x^2 - 1}\right)$$

$$\frac{dy}{dx} = \frac{(x^2 - 1)\frac{d}{dx}(x^2 + 1) - (x^2 + 1)\frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{(x^2 - 1)(2x) - (x^2 + 1)(2x)}{(x^2 - 1)^2}$$

$$= \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

$u = \frac{x - 1}{x + 1}$

Differentiating w.r.t "x", we have

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{x - 1}{x + 1}\right)$$

$$\frac{du}{dx} = \frac{(x + 1)\frac{d}{dx}(x - 1) - (x - 1)\frac{d}{dx}(x + 1)}{(x + 1)^2}$$

$$= \frac{(x + 1)(1) - (x - 1)(1)}{(x + 1)^2}$$

$$= \frac{x + 1 - x + 1}{(x + 1)^2} = \frac{2}{(x + 1)^2}$$

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$ (By Chain rule)

$$= \frac{-4x}{(x^2 - 1)^2} \times \frac{(x + 1)^2}{2} = \frac{-2x(x + 1)^2}{[(x + 1)(x - 1)]^2}$$

$$\frac{dy}{du} = \frac{-2x(x + 1)^2}{(x + 1)^2(x - 1)^2} = \frac{-2x}{(x - 1)^2}$$

Q.3 If $x = \frac{1 - t^2}{1 + t^2}$, $y = \frac{2t}{1 + t^2}$, then prove that

$$y\frac{dy}{dx} + x = 0.$$ *(D.G Khan Board 2017 G-I)*

(Multan Board 2016 G-II)(Sargodha Board 2016)

Sol: $x = \frac{(1 - t^2)}{(1 + t^2)}$

$x = \frac{1 - t^2}{1 + t^2}$, $y = \frac{2t}{1 + t^2}$

Squaring and adding equation, we get

$$x^2 + y^2 = \frac{(1-t^2)^2}{(1+t^2)^2} + \frac{(2t)^2}{(1+t^2)^2}$$

$$x^2 + y^2 = \frac{1+t^4-2t^2+4t^2}{(1+t^2)^2}$$

$$x^2 + y^2 = \frac{1+t^4+2t^2}{1+t^4+2t^2}$$

$$x^2 + y^2 = 1$$

Differentiating w.r.t "x", we have

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$2x + 2y\frac{dy}{dx} = 0$$

Dividing by 2

$$x + y\frac{dy}{dx} = 0$$ Hence proved

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 Derivative of x^3 with respect to x^2 is: *(Sargodha Board 2013)*

(a) 0 (b) Undefined

(c) $\frac{3}{2}x$ (d) $3x^2$

Q.2 $f(x) = \frac{1}{x-2}$, then $f'(2)$= *(D.G.K Board 2015 G-II)*

(a) 1 (b) 0

(c) –1 (d) ∞

Q.3 If $x = a(t - \sin t)$, $y = a(1 + \cos t)$, then $\frac{dy}{dx}$ is given by:

(a) $-\tan\frac{t}{2}$ (b) $\tan\frac{t}{2}$

(c) $-\cot\frac{t}{2}$ (d) $\cot\frac{t}{2}$

Q.4 If $y = \left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)$, $\frac{dy}{dx}$ is equal to:

(a) $3x^2 + \frac{1}{x^2} - 2 - \frac{3}{x^4}$ (b) $2x^2 + \frac{1}{x} - 1 - \frac{3}{x^4}$

(c) $3x^2 + \frac{1}{x^2} - 1 - \frac{3}{x^4}$ (d) None of these

Q.5 Differential of x^6 w.r.t to x^3 is equal to:

(a) $6x^6$ (b) $3x^2$

(c) $2x^3$ (d) x^3

Q.6 If $3x + 4y - 7 = 0$, then $\frac{dy}{dx}$ = *(Bahawalpur Board 2016)(Lahore Board 2014 G-II) (Rawalpindi Board 2019)*

(a) $\frac{3}{4}$ (b) $-\frac{3}{4}$

(c) $\frac{4}{3}$ (d) $-\frac{4}{3}$

Q.7 $\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{du}$ is called: *(Sargodha Board 2013)*

(a) Product rule (b) Power rule

(c) Chain rule (d) Quotient rule

Q.8 If $f(x) = 5x$, $g(x) = x^2$, then $\frac{d}{dx}gof(x)$ = ? *(D.G.K Board 2012 G-I)*

(a) $25x^2$ (b) $50x^2$

(c) $50x$ (d) $\frac{50}{x}$

Q.9 If $x = f(\theta)$, $y = g(\theta)$, then $\frac{dy}{dx}$ = *(Lahore Board 2018 G-II)*

(a) $\frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$ (b) $\frac{dx}{d\theta} \cdot \frac{d\theta}{dy}$

(c) $\frac{d\theta}{dy} \cdot \frac{dx}{d\theta}$ (d) $\frac{dy}{d\theta} \cdot \frac{dx}{d\theta}$

Q.10 If $x = at^2$, $y = 2at$, then $\frac{dy}{dx}$ is equal to. *(Rawalpindi Board 2018)*

(a) t (b) $\frac{1}{t}$

(c) t^2 (d) $\frac{1}{t^2}$

Q.11 $\frac{d}{dx}\left(\frac{1}{g(x)}\right)$ = *(Sahiwal Board 2018)*

(a) $(g(x))^{-2}\, g'(x)$ (b) $-1\,(g'(x))^{-2}\, g(x)$

(c) $(-1)(g'(x))^{-2}\, g'(x)$ (d) $(-1)(g(x))^{-2}\, g'(x)$

Q.12 The derivative of x^3 w.r.t x^2 is equal to: *(Bahawalpur Board 2019)*

(a) $\frac{3x^2}{2}$ (b) $\frac{3x}{2}$

(c) $\frac{2}{3x}$ (d) $\frac{2}{3x^2}$

Q.13 $\frac{d}{dx}(f(u))$ = : *(Faisalabad Board 2019)*

(a) f '(u) (b) f (du)

(c) f '(u) $\frac{du}{dx}$ (d) f '(u) du

EXERCISE 2.5

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find $\frac{dy}{dx}$ if $y = \cos\sqrt{x} + \sqrt{\sin x}$.

(A.J.K Board 2017)(Lahore Board 2016 G-I) (Gujranwala Board 2018

Sol: $y = \cos\sqrt{x} + \sqrt{\sin x}$

Diff. w.r.t x

$$\frac{dy}{dx} = -\sin\sqrt{x}\,\frac{d}{dx}x^{\frac{1}{2}} + \frac{1}{2}(\sin x)^{\frac{-1}{2}}\frac{d}{dx}\sin x$$

$$\frac{dy}{dx} = -\sin\sqrt{x}\left(\frac{1}{2}x^{\frac{-1}{2}}\right) + \frac{1}{2}(\sin x)^{\frac{-1}{2}}\cos x$$

$$= \frac{-\sin\sqrt{x}}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}}$$

Q.2 Find $\frac{dy}{dx}$ if $y = x\cos y$ *(Faisalabad Board 2019 G-II)*

(Gujranwala Board 2011, 2013) (Multan Board 2013 G-II, 2014 G-II) (Sargodha Board 2013 G-I, 2016) (Lahore Board 2017, 2018 G-I, 2019 G-II)

Sol: $\frac{d}{dx}(y) = \frac{d}{dx}(x\cos y)$

Differentiating both sides w.r.t. 'x', we have

$$\frac{dy}{dx} = x\frac{d}{dx}(\cos y) + \cos y\frac{d}{dx}(x)$$

$$= x(-\sin y)\frac{dy}{dx} + \cos y\,(1) = -x\sin y\frac{dy}{dx} + \cos y$$

$$\frac{dy}{dx} + x\sin y\frac{dy}{dx} = \cos y$$

$$(1 + x\sin y)\frac{dy}{dx} = \cos y \Rightarrow \frac{dy}{dx} = \frac{\cos y}{1 + x\sin y}$$

Q.3 If $x = a\cos^3\theta$, $y = b\sin^3\theta$, show that $a\frac{dy}{dx} + b\tan\theta = 0$

(Lahore Board 2011 G-I)(Gujranwala Board 2012) (Multan Board 2018 G-II)

Sol:

Let $x = a\cos^3\theta$	$y = b\sin^3\theta$
$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\cos^3\theta)$	$\frac{dy}{d\theta} = b\frac{d}{d\theta}(\sin^3\theta)$
$= a\left[3\cos^2\theta\frac{d}{d\theta}(\cos\theta)\right]$	$= b\left[3\sin^2\theta\frac{d}{d\theta}(\sin\theta)\right]$
$= a\left[3\cos^2\theta(-\sin\theta)\right]$	$= b\left[3\sin^2\theta\cos\theta\right]$
$= -3a\cos^2\theta\sin\theta$	$= 3b\sin^2\theta\cos\theta$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3b\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{b}{a}\frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow a\frac{dy}{dx} = -b\tan\theta \Rightarrow a\frac{dy}{dx} + b\tan\theta = 0$$

Q.4 Differentiate sin x w.r.t cot x.

(Sahiwal Board 2013)(Rawalpindi Board 2013) (D.G.K Board 2014, 2017 G-II) (Lahore Board 2015 G-II, 2016 G-I) (A.J.K, Rawalpindi Board 2017) (Bahawalpur Board 2018)(Gujranwala Board 2019 G-II)

Ans. Let

$y = \sin x$	$u = \cot x$
Diff. w.r.t 'x'	Diff. w.r.t. 'x'
$\frac{dy}{dx} = \cos x$	$\frac{du}{dx} = -\text{cosec}^2x$

By chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du} \Rightarrow \frac{dy}{du} = \cos x\left(\frac{-1}{\text{cosec}^2x}\right) = -\sin^2x\cos x$$

Q.5 Differentiate $\sin^2x$ w.r.t. $\cos^4x$.

(Bahawalpur Board 2016)(Multan Board 2018 G-II) (Faisalabad, Rawalpindi, Sahiwal Board 2019)

Sol: Let $y = \sin^2x$ and $u = \cos^4x$

Differentiating $y = \sin^2x$ w.r.t. 'x', we have

$$\frac{dy}{dx} = 2\sin x\frac{d}{dx}(\sin x) = 2\sin x\cos x$$

Differentiating $u = \cos^4x$ w.r.t. 'x', we have

$$\frac{du}{dx} = 4\cos^3x\frac{d}{dx}(\cos x) = 4\cos^3x(-\sin x)$$

$$= -4\cos^3x\sin x$$

$$\frac{\frac{dy}{dx}}{\frac{du}{dx}} = \frac{2\sin x\cos x}{-4\cos^3x\sin x} = -\frac{1}{2\cos^2x}$$

Q.6 Prove that $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$, $x \in (-1,1)$.

(Sargodha Board 2013)(Lahore Board 2017 G-II) (Sahiwal Board 2018)(Faisalabad Board 2019 G-II)

Ans. Let, $y = \sin^{-1}x$

$$\sin y = x \qquad y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

Diff. w.r.t 'x'.

$$\cos y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \text{(as required)}$$

Q.7 Prove that $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$

(Faisalabad Board 2019)

Sol: $y = \cos^{-1} x$

$\cos y = x$

Differentiate w.r.t x

$\sin y \frac{dy}{d} = 1$

$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$

Q.8 Prove that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(Gujranwala Board 2016)

Sol: Let $y = \tan^{-1} x \ \Lambda$ (i)

then $x = \tan y \ \Lambda$ (ii) for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Differentiating both sides of (ii) w.r.t. 'x', we have

$1 = \frac{d}{dx}(\tan y) = \frac{d}{dy}(\tan y)\frac{dy}{dx} = \sec^2 y \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$ for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$= \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$ for $x \in R$

Thus $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ for $x \in R$

Q.9 Differentiate w.r.t. 'x': $\sin^{-1}\sqrt{1-x^2}$

(Multan Board 2014 G-I)(Sargodha Board 2019)

Sol: Let $y = \sin^{-1}\sqrt{1-x^2}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{1-x^2}\right)$

$= \frac{1}{\sqrt{1-\left(1-x^2\right)}} \cdot \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \frac{d}{dx}\left(1-x^2\right)$

$= \frac{1}{\sqrt{1-1+x^2}} \cdot \frac{1}{2} \frac{1}{\sqrt{1-x^2}} (0-2x)$

$= \frac{-x}{\sqrt{x^2}\left(\sqrt{1-x^2}\right)} = \frac{-x}{x\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-x^2}}$

Q.10 If $\tan y(1+\tan x) = 1 - \tan x$, show that $\frac{dy}{dx} = -1$.

(Rawalpindi Board 2013)

Ans. $\tan y(1+\tan x) = 1 - \tan x$

$\tan y = \frac{1-\tan x}{1+\tan x}$

$\tan y = \frac{\tan\frac{\pi}{4} - \tan x}{1+\tan x \tan\frac{\pi}{4}}$ $\quad \because \tan\frac{\pi}{4} = 1$

$\tan y = \tan\left(\frac{\pi}{4} - x\right)$

$y = \frac{\pi}{4} - x$

Diff. w.r.t 'x'.

$\frac{dy}{dx} = -1$

Q.11 Differentiate w.r.t. 'x': $\cot^{-1}\left(\frac{x}{a}\right)$

(Sahiwal Board 2014 G-II)(Lahore Board 2012 G-I)
(D.G Khan Board 2017 G-I)(Faisalabad Board 2013)
(Gujranwala Board 2019 G-II)

Sol: Let $y = \cot^{-1}\left(\frac{x}{a}\right)$ Differentiating w.r.t. 'x', we have

$\frac{dy}{dx} = -\frac{1}{1+\left(\frac{x}{a}\right)^2} \times \frac{d}{dx}\left(\frac{x}{a}\right) = -\frac{1}{\frac{a^2+x^2}{a^2}} \cdot \frac{1}{a}$

$= -\frac{a^2}{a^2+x^2} \times \frac{1}{a} = -\frac{a}{a^2+x^2}$

Q.12 If $y = \text{Tan}\left(p \text{ Tan}^{-1} x\right)$, show that

$\left(1+x^2\right) y_1 - p\left(1+y^2\right) = 0$

(Multan Board 2014 G-I)
(Lahore Board 2012 G-II, 2015 G-I)

Sol: $y = \text{Tan}\left(p \text{ Tan}^{-1} x\right)$

$\text{Tan}^{-1} y = p \text{ Tan}^{-1} x$

$\frac{d}{dx}\left(\text{Tan}^{-1} y\right) = \frac{d}{dx}\left(p \text{ Tan}^{-1} x\right)$

$\frac{1}{1+y^2}\frac{dy}{dx} = p\frac{1}{1+x^2}$

$\left(1+x^2\right)\frac{dy}{dx} = p\left(1+y^2\right)$

$\left(1+x^2\right) y_1 = p\left(1+y^2\right)$

$\left(1+x^2\right) y_1 - p\left(1+y^2\right) = 0$

Q.13 Show that $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$.

(Gujranwala Board 2019)(Lahore Board 2019 G-II)
(Rawalpindi Board 2019)

Sol: $y = \cot^{-1} x$

$\cot y = x$

Diff. w.r.t x

$-\text{cosec}^2 y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{-1}{\text{cosec}^2 y}$

$= \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}$

Q.14 If $y = \sin^{-1}\frac{x}{a}$ then $\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$.

(Gujranwala Board 2019)

Sol: $y = \sin^{-1}\frac{x}{a}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \frac{d}{dx}\frac{x}{a} = \frac{1}{\frac{\sqrt{a^2-x^2}}{a}} - \frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}}$$

LONG QUESTIONS

Q.1 Differentiate ab-intio with respect to x, $y = \sin\sqrt{x}$.

(Faisalabad Board 2013)(Sahiwal Board 2014)

Sol: Let $f(x) = \sin\sqrt{x}$ and $f(x+\delta x) = \sin\sqrt{x+\delta x}$

$$f'(x) = \lim_{\delta x \to 0}\frac{f(x+\delta x)-f(x)}{\delta x} = \lim_{\delta x \to 0}\frac{\sin\sqrt{x+\delta x}-\sin\sqrt{x}}{\delta x}$$

$$= \lim_{\delta x \to 0}\frac{2\cos\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{x+\delta x-x} = \lim_{\delta x \to 0}\frac{2\cos\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}.\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\left(\sqrt{x+\delta x}\right)^2-\left(\sqrt{x}\right)^2}$$

$$= \lim_{\delta x \to 0}\frac{2\cos\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}.\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\left[\sqrt{x+\delta x}+\sqrt{x}\right]\left[\sqrt{x+\delta x}-\sqrt{x}\right]} = \lim_{\delta x \to 0}\frac{2\cos\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{\sqrt{x+\delta x}+\sqrt{x}} \times \frac{\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\sqrt{x+\delta x}-\sqrt{x}}$$

$$= \lim_{\delta x \to 0}\frac{\cos\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{\sqrt{x+\delta x}+\sqrt{x}} \times \frac{\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

$$= \lim_{\delta x \to 0}\frac{\cos\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{\sqrt{x+\delta x}+\sqrt{x}} \times \lim_{\delta x \to 0}\frac{\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

$$= \frac{\cos\frac{\sqrt{x+0}+\sqrt{x}}{2}}{\sqrt{x+0}+\sqrt{x}} \times 1 = \frac{\cos\frac{\sqrt{x}+\sqrt{x}}{2}}{\sqrt{x}+\sqrt{x}} = \frac{\cos\frac{2\sqrt{x}}{2}}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

Q.2 Differentiate $\cos\sqrt{x}$ from the first principles.

(Lahore Board 2015 G-I)

Sol: Let $f(x) = \cos\sqrt{x}$

$\Rightarrow f(x+\delta x) = \cos\sqrt{x+\delta x}$

$$f'(x) = \frac{f(x+\delta x)-f(x)}{\delta x} = \lim_{\delta x \to 0}\frac{\cos\sqrt{x+\delta x}+\cos\sqrt{x}}{\delta x}$$

$$= \lim_{\delta x \to 0}\frac{-2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\delta x} = \lim_{\delta x \to 0}\frac{-2\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{(x+\delta x)-x}$$

$$= \lim_{\delta x \to 0}\frac{-2\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{(\sqrt{x+\delta x})^2-(\sqrt{x})^2} = \lim_{\delta x \to 0}\frac{-2\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{(\sqrt{x+\delta x}+\sqrt{x})(\sqrt{x+\delta x}-\sqrt{x})}$$

$$= \lim_{\delta x \to 0}\frac{\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{(\sqrt{x+\delta x}+\sqrt{x})}\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)} = \lim_{\delta x \to 0}\frac{\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{\sqrt{x+\delta x}+\sqrt{x}} \times \lim_{\delta x \to 0}\frac{\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

$$= \frac{\sin\frac{\sqrt{x+0}+\sqrt{x}}{2}}{\sqrt{x}+\sqrt{x}} \times 1 = \frac{-\sin\frac{2\sqrt{x}}{2}}{2\sqrt{x}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Q.3 If $x = a\cos^3\theta$, $y = b\sin^3\theta$, show that

$$a\frac{dy}{dx} + b\tan\theta = 0$$ *(D.G.K Board 2014 G-I)*

Sol: See Short Question 8

Q.4 If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots \infty}}}$, then prove that $(2y-1)\frac{dy}{dx} = \sec^2 x$.

(D.G Khan, Rawalpindi Board 2017 G-II)

Sol: $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots \infty}}}$

Squaring both sides, we have

$y^2 = \tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots \infty}}$

$y^2 = \tan x + y$

Differentiating w.r.t "x", we have

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x + y)$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x) + \frac{d}{dx}(y)$$

$$2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \Rightarrow 2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$\frac{dy}{dx}(2y - 1) = \sec^2 x \qquad \text{Hence proved}$$

Q.5 If $y = \tan(\tan^{-1} x)$, show that $(1 + x^2) y_1 - p(1 + y^2) = 0$

(Lahore Board 2017 G-II)

Sol: See Short Question 25

Q.6 Differentiate with respect to x $\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$.

(Sargodha Board 2013)

Sol: Let $y = \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Differentiating w.r.t "x", we have

$$\frac{dy}{dx} = \frac{d}{dx}\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right) \quad \Theta \ \frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2 - 1}} \cdot \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{(x^2-1)}{(x^2+1)\sqrt{\frac{(x^2+1)^2}{(x^2-1)^2}-1}} \cdot \left(\frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}\right)$$

$$= \frac{(x^2-1)}{(x^2+1)\sqrt{\frac{(x^2+1)^2}{(x^2-1)^2}-1}} \cdot \left(\frac{(x^2-1)2x - (x^2+1)2x}{(x^2-1)^2}\right)$$

$$= \frac{(x^2-1)}{(x^2+1)\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2-1)^2}}} \left(\frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2}\right)$$

$$= \frac{(x^2-1)}{(x^2+1)\sqrt{\frac{4(x^2)(1)}{(x^2-1)^2}}} \left(\frac{-4x}{(x^2-1)^2}\right)$$

$\Theta\ (a+b)^2 - (a-b)^2 = 4ab$

$$= \frac{(x^2-1)}{(x^2+1)\frac{2x}{x^2-1}} \left(\frac{-4x}{(x^2-1)^2}\right)$$

$$= \frac{(x^2-1)(x^2-1)}{(x^2+1)2x} \left(\frac{-4x}{(x^2-1)^2}\right)$$

$$\frac{dy}{dx} = \frac{-2}{x^2+1}$$

Q.7 Find $\frac{dy}{dx}$ of $x = a(\cos t + \sin t)$, $y = a(\sin t - t\cos t)$

(Lahore Board 2013 G-II)(Lahore Board 2017 G-I)

Sol: $x = a(\cos t + \sin t)$ ... (i)

$y = a(\sin t - t\cos t)$ ... (ii)

Differentiating w.r.t "t", we have

$$\frac{dx}{dt} = a\frac{d}{dt}(\cos t + \sin t)$$

$$= a\left[\frac{d}{dt}(\cos t) + \frac{d}{dt}(\sin t)\right]$$

$$\frac{dx}{dt} = a(-\sin t + \cos t)$$

$$\frac{dx}{dt} = a(\cos t - \sin t)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \text{... (iii)} \qquad \text{(By chain rule)}$$

Differentiating w.r.t "t", we have

$$\frac{dy}{dt} = a\frac{d}{dt}(\sin t - t\cos t)$$

$$= a\left[\frac{d}{dt}(\sin t) - \frac{d}{dt}(t\cos t)\right]$$

$$\frac{dy}{dt} = a\left(\cos t - \left(t\frac{d}{dt}(\cos t) + \cos t\frac{dt}{dt}\right)\right)$$

$$\frac{dy}{dt} = a(\cos t - (t(-\sin t) + \cos t))$$

$$= a(\cos t + t\sin t - \cos t) = a\,t\sin t$$

Putting values in equation (iii)

$$\frac{dy}{dx} = a\,t\sin t\left(\frac{1}{a(\cos t - \sin t)}\right) = \frac{t\sin t}{\cos t - \sin t}$$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 $\frac{d}{dx}(-\cot x)$ equals: *(Sahiwal Board 2014)*

(Bahawalpur Board 2018)

(a) $\sec^2 x$ (b) $\operatorname{cosec}^2 x$

(c) $-\operatorname{cosec}^2 x$ (d) $-\sec^2 x$

Q.2 If $f(x) = \sin x$, then $f'(\pi) =$ _____

(D.G.K Board 2014 G-II)

(a) -1 (b) 0

(c) 1 (d) $\frac{1}{2}$

Q.3 $\frac{d}{dx}\sin^{-1}x$ =

(Multan Board 2013 G-II, D.G.K Board 2014 G-I)

(a) $\cos^{-1}x$ (b) $\frac{1}{\sqrt{1-x^2}}$

(c) $\frac{-1}{\sqrt{1-x^2}}$ (d) $\frac{1}{\sqrt{1+x^2}}$

Q.4 If $f(x)=\frac{1}{\sqrt{1+x^2}}$, the $f'(\cot x)$ = ?

(D.G.K Board 2012 G-I)

(a) $\frac{d}{dx}(-\cos x)$ (b) $\frac{d}{dx}(-\sin x)$

(c) $\frac{d}{dx}\cos x$ (d) $\frac{d}{dx}\sin x$

Q.5 If $f(x)=\tan^{-1}x$, then $f'(\cot x)$ = _____

(Multan Board 2014 G-II)

(a) $\frac{1}{1+x^2}$ (b) $\sin^2 x$

(c) $\cos^2 x$ (d) $\sec^2 x$

Q.6 $\frac{d}{dx}\left(x-\frac{\cos 2x}{2}\right)$ is equal to:

(a) $\sin x-\cos x$ (b) $(\sin x-\cos x)^2$

(c) $\sin x-\cos x$ (d) $(\sin x+\cos x)^2$

Q.7 $\frac{d}{dx}\left(\frac{1}{\sec x}\right)$ is equal to: *(Rawalpindi Board 2014)*

(a) $\frac{d}{dx}\sin x$ (b) $\frac{d}{dx}\operatorname{cosec} x$

(c) $\frac{d}{dx}\cos x$ (d) $\frac{d}{dx}\cot x$

Q.8 $\frac{1}{2}\frac{d}{dx}[\tan^{-1}x-\cot^{-1}x]$ = *(Rawalpindi Board 2016)*

(Multan Board 2018 G-I)

(a) $-\frac{1}{1+x^2}$ (b) $\frac{1}{1+x^2}$

(c) $\frac{1}{1-x^2}$ (d) $\frac{-1}{1-x^2}$

Q.9 $\frac{d}{dx}\cot^{-1}x$ = *(Sahiwal Board 2013)*

(a) $\frac{1}{1+x^2}$ (b) $\frac{1}{1-x^2}$

(c) $\frac{-1}{1-x^2}$ (d) $\frac{-1}{1+x^2}$

Q.10 $\frac{d}{dx}\sin x^2$ equals: *(D.G.K Board 2011)*

(a) $\cos x^2$ (b) $-\cos x^2$

(c) $2x\cos x^2$ (d) $-2x\cos x^2$

Q.11 Derivative of $-\sin x$ w.r.t $\sin x$ is:

(Multan Board 2013 G-II)

(a) $\cos x$ (b) $-\cos x$

(c) 1 (d) -1

Q.12 $\frac{d}{dx}(\tan^{-1}x+\cot^{-1}x)$ =

(Sargodha Board 2013, Rawalpindi Board 2013)

(a) 1 (b) -1

(c) 0 (d) 2

Q.13 $\frac{d}{dx}(\operatorname{cosec}^{-1}x)$ is equal to: *(Rawalpindi Board 2014)*

(a) $\frac{1}{x\sqrt{x^2-1}}$ (b) $\frac{-1}{x\sqrt{x^2-1}}$

(c) $\frac{1}{1+x^2}$ (d) $\frac{-1}{1+x^2}$

Q.14 $\frac{d}{dx}(\sin\sqrt{x})$ is equal to:*(Rawalpindi Board 2013)*

(a) $\cos\sqrt{x}$ (b) $-\frac{\cos\sqrt{x}}{2\sqrt{x}}$

(c) $\frac{\cos\sqrt{x}}{2\sqrt{x}}$ (d) $\frac{\cos\sqrt{x}}{\sqrt{x}}$

Q.15 If $y=\cos\sqrt{x}$, then $\frac{dy}{dx}$ =

(Faisalabad Board 2013)

(a) $-\sin\sqrt{x}$ (b) $\frac{\cos\sqrt{x}}{\sqrt{x}}$

(c) $-\frac{\sin\sqrt{x}}{\sqrt{x}}$ (d) $-\frac{\sin\sqrt{x}}{2\sqrt{x}}$

Q.16 f(x)=cot x, then $f'\left(\frac{\pi}{6}\right)$ = *(D.G.K Board 2015 G-II)*

(a) -4 (b) 4

(c) $\frac{1}{4}$ (d) $-\frac{1}{4}$

Q.17 $f(x)=\sec^{-1}x$, then $f'(\sec x)$ =

(D.G.K Board 2015 G-II)

(a) $\frac{1}{x\sqrt{x^2-1}}$ (b) $\sec x\tan x$

(c) $\cos^2 x\operatorname{cosec} x$ (d) $-\cos^2 x\operatorname{cosec} x$

Q.18 If $f(x)=\sin x$, then $f'(0)$ =

(D.G.K Board 2015 G-I)(Rawalpindi Board 2019)

(a) $\cos x$ (b) 1

(c) 0 (d) -1

Q.19 $\frac{d}{dx}\tan^{-1}x$ = *(D.G.K Board 2015 G-I)*

(a) $\frac{1}{1-x^2}$ (b) $\frac{1}{1+x^2}$

(c) $\tan x$ (d) $\sec x$

Q.20 If $f(x) = \tan x$, then $f'(x) = ?$ *(D.G.K Board 2013 G-II)*

(a) $\sec^2 x$ (b) $\sec^2 x$
(c) $\operatorname{cosec}^2 x$ (d) $-\operatorname{cosec}^2 x$

Q.21 If $y = \sin^{-1} x$, then $\frac{dy}{dx} = ?$ *(D.G.K Board 2013 G-II) (Faisalabad Board 2018)(Lahore Board 2018 G-I) (D.G.K Board 2012 G-II)*

(a) $\frac{1}{\sqrt{1+x^2}}$ (b) $\frac{-1}{\sqrt{1-x^2}}$
(c) $\frac{1}{\sqrt{1-x^2}}$ (d) $\frac{-1}{\sqrt{x^2-1}}$

Q.22 $\frac{d}{dx}\sec x =$ *(D.G.K Board 2012 G-II)*

(a) $\tan x$ (b) $\sin x$
(c) $\sec x \tan x$ (d) $\sec x$

Q.23 $\frac{d}{dx}(\sec^{-1} x + \operatorname{cosec}^{-1} x)$ *(Faisalabad Board 2018)*

(a) 1 (b) −1
(c) 0 (d) 2

Q.24 If $y = \sin u$, $u = \sin^{-1} x^3$, then $\frac{dy}{dx} = ?$ *(D.G.K Board 2012 G-II)*

(a) $\cos x^3$ (b) $3x^2$
(c) x^3 (d) $\sin x^3$

Q.25 $\frac{d}{dx}\left[x - \frac{\sin 2x}{2}\right]$ is equal to: *(D.G.K Board 2011)*

(a) $2\sin^2 x$ (b) $2\cos^2 x$
(c) $-2\sin^2 x$ (d) $-2\cos^2 x$

Q.26 $\frac{d}{dx}\cot^{-1} x$ equals to: *(D.G.K Board 2010) (A.J.K Board 2017)*

(a) $\frac{1}{1+x^2}$ (b) $\frac{1}{\sqrt{1+x^2}}$
(c) $-\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{-1}{1+x^2}$

Q.27 $\frac{d}{dx}(\sin x) =$

(a) $\sin x$ (b) $\cos x$
(c) $-\sin x$ (d) $-\cos x$

Q.28 $\frac{d}{dx}(\tan x) =$

(a) $\sec x \tan x$ (b) $\sec^2 x$
(c) $-\sec x \tan x$ (d) $-\sec^2 x$

Q.29 $\frac{d}{dx}(\operatorname{cosec} x) =$

(a) $\operatorname{cosec} x \cot x$ (b) $\operatorname{cosec}^2 x$
(c) $-\operatorname{cosec} x \cot x$ (d) $-\operatorname{cosec}^2 x$

Q.30 $\frac{d}{dx}(\sec x) =$ *(Rawalpindi Board 2019)*

(a) $\sec x \tan x$ (b) $-\sec^2 x$
(c) $-\sec x \tan x$ (d) $\sec^2 x$

Q.31 $\frac{d}{dx}(\cot x) =$

(a) $\operatorname{cosec}^2 x$ (b) $-\operatorname{cosec}^2 x$
(c) $-\operatorname{cosec} x \cot x$ (d) $\operatorname{cosec} x \cot x$

Q.32 $\frac{d}{dx}(\sin^{-1} x) =$ *(Multan Board 2013 G-I)*

(a) $\frac{1}{\sqrt{x^2-1}}$ (b) $\frac{1}{\sqrt{1-x^2}}$
(c) $\frac{1}{\sqrt{1-x^2}}$ (d) $-\frac{1}{\sqrt{x^2-1}}$

Q.33 $\frac{d}{dx}(\cos^{-1} x) =$

(a) $\frac{1}{\sqrt{x^2-1}}$ (b) $\frac{1}{\sqrt{1-x^2}}$
(c) $-\frac{1}{\sqrt{1-x^2}}$ (d) $-\frac{1}{\sqrt{x^2-1}}$

Q.34 $\frac{d}{dx}(\operatorname{cosec}^{-1} x) =$

(a) $-\frac{1}{|x|\sqrt{x^2-1}}$ (b) $\frac{1}{|x|\sqrt{x^2-1}}$
(c) $-\frac{1}{|x|\sqrt{1-x^2}}$ (d) $\frac{1}{|x|\sqrt{1-x^2}}$

Q.35 $\frac{d}{dx}(\sec^{-1} x) =$

(a) $-\frac{1}{|x|\sqrt{x^2-1}}$ (b) $\frac{1}{|x|\sqrt{x^2-1}}$
(c) $-\frac{1}{|x|\sqrt{1-x^2}}$ (d) $\frac{1}{|x|\sqrt{1-x^2}}$

Q.36 $\frac{d}{dx}(\tan^{-1} x) =$ *(Lahore Board 2012 G-I)*

(a) $\frac{1}{1+x^2}$ (b) $-\frac{1}{1+x^2}$
(c) $\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{1}{\sqrt{1-x^2}}$

Q.37 If $f(x) = \sin x$, then slope of tangent line at $x = 0$ is: *(Gujranwala Board 2014)*

(a) 0 (b) 1
(c) −1 (d) Does not exist

Q.38 $\frac{d}{dx}\cos x =$ *(Lahore Board 2014 G-II)*

(a) $-\cos x$ (b) $\sec^2 x$
(c) $-\sin x$ (d) $\sec^2 x$

Q.39 The derivative of $\cos x$ w.r.t $\cos x$ is: *(Gujranwala Board 2013)*

(a) 0 (b) 1
(c) $\sin x$ (d) $\cos x$

Q.40 $\frac{d}{dx}\left(\sqrt{\tan x}\right) =$ ——— *(Gujranwala Board 2013)*
(Bahawalpur Board 2018)

(a) $\frac{\sec^2 x}{2\sqrt{\tan x}}$ (b) $\frac{1}{2}\sqrt{\tan x}\sec^2 x$

(c) $\frac{1}{2}\tan^{3/2} x \cdot \sec^2 x$ (d) $\frac{\sec^2 x}{\sqrt{\tan x}}$

Q.41 $\frac{d}{dx}\sin^{-1}\frac{x}{a} =$ ——— *(Gujranwala Board 2013)*

(a) $\frac{a}{\sqrt{a^2 - x^2}}$ (b) $\frac{-a}{\sqrt{a^2 - x^2}}$

(c) $\frac{-1}{\sqrt{a^2 - x^2}}$ (d) $\frac{1}{\sqrt{a^2 - x^2}}$

Q.42 $\frac{d}{dx}\cos x°$ *(Lahore Board 2013 G-II)*

(a) $-\sin x°$ (b) $x° \sin x°$

(c) $\frac{\pi}{180}\sin x°$ (d) $-\frac{\pi}{180}\sin x°$

Q.43 $\frac{d}{dx}(\text{cosec}^2 x - \cot^2 x)$ is: *(Lahore Board 2013 G-I)*

(a) $\cot^2 x + \text{cosec}^2 x$
(b) $-2\text{cosec } x \cot x + 2 \cot x \text{ cosec}^2 x$
(c) 0
(d) $\sec^2 x + \tan^2 x$

Q.44 $\frac{d}{dx}\cos^2 x$ is: *(Lahore Board 2013 G-I)*

(a) $-\sin^2 x$ (b) $-2\cos x \sin x$
(c) $\sin 2x$ (d) $\sec^2 x$

Q.45 The derivative of tan x with respect to tan x is: *(Gujranwala Board 2012)*

(a) Zero (b) 1
(c) $\sec^2 x$ (d) $2\sec^2 x$

Q.46 If $f(x) = \sin x$, then $f'(\cos^{-1} x) =$ *(Gujranwala Board 2012)*

(a) $\cos x$ (b) $\sin x$
(c) $-x$ (d) x

Q.47 If $\frac{d}{dx} f(x) = \frac{x}{\sqrt{1 - x^2}}$, then $f'(\sin x) =$
(Lahore Board 2012 G-I)

(a) $\sec^{-1} x$ (b) $\frac{1}{\cot x}$

(c) $\frac{1}{\tan x}$ (d) $\tan^{-1} x$

Q.48 If $y = \sec\left(\frac{3\pi}{2} - x\right)$, then y_1 equals:
(Lahore Board 2015 G-II)

(a) $\text{cosec } x \cot x$ (b) $-\text{cosec } x \cot x$
(c) $\sec x \tan x$ (d) $-\sec x \tan x$

Q.49 If $f(x) = \cos x$, then $f'(0)$ is equal to:
(Lahore Board 2015 G-I)

(a) 0 (b) -1
(c) 1 (d) $\frac{1}{2}$

Q.50 If $\sqrt{\cot x} = y$, then $\frac{dy}{dx} =$ *(Bahawalpur Board 2016)*

(a) $\frac{\text{cosec}^2 x}{2\sqrt{\cot x}}$ (b) $-\frac{\text{cosec}^2 x}{2\sqrt{\cot x}}$
(c) $\frac{2\sqrt{\cot x}}{\text{cosec}^2 x}$ (d) $\frac{-2\sqrt{\cot x}}{\text{cosec}^2 x}$

Q.51 $\frac{d}{dx}(-\cot x) =$ *(Faisalabad Board 2016)*

(a) $-\text{cosec } x \cot x$ (b) $\text{cosec } x \cot x$
(c) $\text{cosec}^2 x$ (d) $-\text{cosec}^2 x$

Q.52 If $f(x) = \tan x$, then $f'\left(\frac{\pi}{4}\right) =$
(Gujranwala Board 2016)

(a) 1 (b) $\frac{1}{2}$
(c) 2 (d) $\frac{1}{\sqrt{2}}$

Q.53 $\frac{d}{dx}\left[\frac{1}{\sin x}\right] =$ *(Rawalpindi Board 2016)*

(a) $\frac{1}{\cos x}$ (b) $-\frac{\sin x}{\cos x}$
(c) $\text{cosec}^2 x$ (d) $-\text{cosec cot } x$

Q.54 $\frac{d}{dx}(\cos x^2) =$ *(Sargodha Board 2016)*
(Gujranwala Board 2018)(Lahore, Sahiwal Board 2019)

(a) $-2x \sin x^2$ (b) $2x \sin x^2$
(c) $-\frac{\sin x^2}{2x}$ (d) $-2 \sin x \cos x$

Q.55 $\frac{d}{dx}\sqrt{\tan x} =$ *(Faisalabad Board 2017)*

(a) $\frac{\sec^2 x}{2\sqrt{\tan x}}$ (b) $\frac{\sec^2 x}{\sqrt{\tan x}}$
(c) $\frac{\sec x}{\sqrt{\tan x}}$ (d) $\frac{\sqrt{\sec x}}{\tan x}$

Q.56 $\frac{d}{dx}\cot^2 2x$ *(Lahore Board 2017 G-I)*

(a) $4 \cot 2x \text{ cosec } 2x$ (b) $-4 \cot 2x \text{ cosec}^2 2x$
(c) $4 \cot^2 2x \text{ cosec } 2x$ (d) $-4 \cot 2x$

Q.57 If $f(x) = \cos x$, then $f'\left(\frac{\pi}{2}\right) =$
(Lahore Board 2017 G-I)

(a) 1 (b) 0
(c) $\frac{1}{2}$ (d) -1

Q.58 If $y = \cos x$, $u = \sin x$, then $\frac{dy}{dx} =$

(Lahore Board 2017 G-II)(Multan Board 2017 G-I)

(a) $\cos x$ (b) $-\cot x$

(c) $-\tan x$ (d) $-\operatorname{cosec} x$

Q.59 $\frac{d}{dx}\left[\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}\right] =$

(Multan Board 2017 G-I)

(a) 1 (b) $\frac{1}{2}$

(c) 0 (d) -1

Q.60 $\frac{d}{dx} \sec x =$ *(Sahiwal Board 2017)*

(a) $\sec x \tan x$ (b) $-\sec x \tan x$

(c) $\sec^2 x$ (d) $\sec x \tan^2 x$

Q.61 If $f(x) = \tan^{-1}x$, then $f'(\cot x) =$

(Rawalpindi Board 2016)

(a) $\cos^2 x$ (b) $\sin^2 x$

(c) $\operatorname{cosec}^2 x$ (d) $\cot^2 x$

Q.62 $\frac{1}{x}\frac{d}{dx}(\sin x^2) =$

(a) $2x \cos x^2$ (b) $2\cos x^2$

(c) $2x \cos^2 x$ (d) $2 \cos^2 x$

Q.63 $\frac{1}{1+x^2}$ is derivative of. *(A.J.K Board 2017)*

(a) $\sin^{-1} x$ (b) $\sec^{-1} x$

(c) $\tan^{-1} x$ (d) $\cot^{-1} x$

Q.64 $\frac{d}{dx}(-\operatorname{cosec} x) =$ *(A.J.K Board 2017)*

(a) $\cot^2 x$ (b) $\operatorname{cosec}^2 x$

(c) $\tan x \operatorname{cosec} x$ (d) $\operatorname{cosec} x \cot x$

Q.65 $\frac{d}{dx}\cot^{-1}\frac{x}{a} =$ *(Multan Board 2018 G-II)*

(a) $\frac{a}{a^2+x^2}$ (b) $\frac{a^2}{a^2+x^2}$

(c) $\frac{-a}{a^2+x^2}$ (d) $\frac{-1}{a^2+x^2}$

Q.66 $\frac{d}{dx}(\cos^{-1}3x) =$

(Multan Board 2018 G-II)(Gujranwala Board 2019)

(a) $\frac{3}{\sqrt{1-9x^2}}$ (b) $\frac{-3}{\sqrt{1-9x^2}}$

(c) $\frac{1}{\sqrt{1-9x^2}}$ (d) $\frac{-1}{\sqrt{1-9x^2}}$

Q.67 If $\int f(x)\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$, then $f(x) =$

(Sahiwal Board 2018)

(a) $\frac{1}{\sqrt{x^2-a^2}}$ (b) $\frac{1}{x\sqrt{x^2-a^2}}$

(c) $\frac{1}{x\sqrt{x^2+a^2}}$ (d) $\frac{1}{x\sqrt{a^2-x^2}}$

Q.68 $\frac{d}{dx}(\cos^2 x - \sin^2 x) =$ *(Sargodha Board 2018)*

(a) $2 \cos 2x$ (b) $-2 \cos 2x$

(c) $-2 \sin 2x$ (d) $2 \sin 2x$

Q.69 If $f(x) = \tan x$, then $f'\left(\frac{\pi}{6}\right) =$

(Sargodha Board 2018)

(a) $\frac{4}{3}$ (b) $\frac{3}{4}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}}{4}$

Q.70 If $y = \sin^{-1}\frac{x}{a}$, the $\sin y =$: *(Lahore Board 2019)*

(a) $\cos y$ (b) $\cos x$

(c) $\frac{x}{a}$ (d) $\frac{y}{a}$

Q.71 If $f(x) = \sin x$, then $f'(0) =$

(Rawalpindi Board 2019)

(a) 0 (b) 1

(c) -1 (d) 2

Q.72 The derivative of Cot x w.r.t x equals

(Sargodha Board 2019)

(a) $-\operatorname{Cosec}^2 x$ (b) $\operatorname{Cosec}^2 x$

(c) $-\operatorname{Sec}^2 x$ (d) $\operatorname{Sec}^2 x$

EXERCISE 2.6

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find $f'(x)$ if $f(x) = e^{\sqrt{x}-1}$

(Lahore Board 2012 G-II, 2013 G-II)

(D.G Khan Board 2017)(Sahiwal Board 2018)

Sol: Differentiating w.r.t 'x', we have

$$f'(x) = e^{\sqrt{x}-1} \times \frac{d}{dx}\left(\sqrt{x}-1\right)$$

$$= e^{\sqrt{x}-1}\left(\frac{1}{2\sqrt{x}} - 0\right) = \frac{1}{2\sqrt{x}}\, e^{\sqrt{x}-1}$$

Q.2 Find $\frac{dy}{dx}$ if $y = \lambda n (9 - x^2)$

(Multan Board 2017 G-I)(Sargodha Board 2019)

Sol: Differentiating w.r.t. 'x', we have

$$\frac{dy}{dx} = \frac{1}{9-x^2} \times \frac{d}{dx}(9 - x^2) = \frac{1}{9-x^2} \times (0 - 2x)$$

$$= \frac{-2x}{9-x^2}$$

Q.3 If $y = x^2 \ln \frac{1}{x}$ find $\frac{dy}{dx}$.

(Gujranwala Board 2013, 2017)

Ans. $y = x^2 \ln \frac{1}{x}$

$y = x^2 \ln x^{-1} = -x^2 \ln x$

Differentiate w.r.t 'x'.

$$\frac{dy}{dx} = \frac{-d}{dx}(x^2 \ln x)$$

$$\frac{dy}{dx} = \left[x^2\left(\frac{1}{x}\right) + 2x \ln x\right]$$

$$\frac{dy}{dx} = -(x + 2x \ln x)$$

$$\frac{dy}{dx} = -x(1 + 2x \ln x)$$

Q.4 ind f'(x) if $f(x) = \frac{e^x}{e^{-x}+1}$

(Rawalpindi Board 2014, 2019) (Sahiwal Board 2019)

Sol: Differentiating w.r.t. 'x', we have

$$f'(x) = \frac{(e^{-x}+1)\frac{d}{dx}(e^x) - e^x \frac{d}{dx}(e^{-x}+1)}{(e^{-x}+1)^2}$$

$$= \frac{(e^{-x}+1)e^x - e^x\{e^{-x}(-1)+0\}}{(e^{-x}+1)^2}$$

$$= \frac{e^{-x}e^x + e^x + e^x e^{-x}}{(e^{-x}+1)^2} = \frac{e^{-x} + x + e^x + e^{x-x}}{(e^{-x}+1)^2}$$

$$= \frac{e^0 + e^x + e^0}{(e^{-x}+1)^2}$$

$$= \frac{1 + e^x + 1}{(e^{-x}+1)^2} = \frac{2 + e^x}{(e^{-x}+1)^2}$$

Q.5 If $y = x\, e^{\sin x}$ then find $\frac{dy}{dx}$

(Lahore Board 2013)(Bahawalpur Board 2016, 2019)
(D.G.K Board 2014)(Sargodha Board 2017)
(Lahore Board 2018 G-I)(Sargodha Board 2018)

Ans. $y = x\, e^{\sin x}$

Diff. w.r.t. 'x'

$$\frac{dy}{dx} = x\frac{d}{dx}e^{\sin x} + e^{\sin x}\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = xe^{\sin x}.\cos x + e^{\sin x}$$

$$\frac{dy}{dx} = e^{\sin x}(1 + x\cos x)$$

Q.6 Find $\frac{dy}{dx}$ if $y = \frac{x}{\ln x}$ *(Rawalpindi Board 2016)*

(Sahiwal Board 2014 G-II)(Faisalabad Board 2017)
(Bahawalpur Board 2016, 2019)
(Gujranwala Board 2019 G-II)

Sol: Differentiating w.r.t. 'x', we have

$$\frac{dy}{dx} = \frac{\ln x \frac{d}{dx}(x) - x\frac{d}{dx}(\ln x)}{(\ln x)^2} = \frac{\ln x\,(1) - x\left(\frac{1}{x}\right)}{(\ln x)^2}$$

$$= \frac{\ln x - 1}{(\ln x)^2}$$

Q.7 Find $\frac{dy}{dx}$ if $y = x^2 \lambda n \frac{1}{x}$

(Gujranwala Board 2013, 2016)

Sol: Differentiating w.r.t. 'x', we have

$$\frac{dy}{dx} = x^2\frac{d}{dx}\left(\ln\frac{1}{x}\right) + \lambda n\frac{1}{x}\frac{d}{dx}(x^2)$$

$$= x^2\frac{1}{\frac{1}{x}}\cdot\frac{d}{dx}\left(\frac{1}{x}\right) + \lambda n\frac{1}{x}(2x)$$

$$= x^2 \cdot x\left(-\frac{1}{x^2}\right) + 2x\,\lambda n\frac{1}{x}$$

$$= -x + 2x\,\lambda n\frac{1}{x} = x\left\{2\ln\frac{1}{x} - 1\right\}$$

Q.8 Find $\frac{dy}{dx}$ if $y = \lambda n\left(x + \sqrt{x^2+1}\right)$

(Sahiwal Board 2013) (Bahawalpur Board 2018)

Sol: Differentiating w.r.t. 'x', we have

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}} \times \frac{d}{dx}\left(x + \sqrt{x^2+1}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}} \times \left\{1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}}\frac{d}{dx}(x^2+1)\right\}$$

$$= \frac{1}{x+\sqrt{x^2+1}} \times \left\{1 + \frac{1}{2}\frac{1}{\sqrt{x^2+1}}(2x+0)\right\}$$

$$= \frac{1}{x+\sqrt{x^2+1}} \times \left(1 + \frac{2x}{2\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}} \times \left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}} \times \left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$

Q.9 Find $\frac{dy}{dx}$ if $y = e^{-x}(x^3+2x^2+1)$

(Lahore Board 2016 G-I)

Sol: Differentiating w.r.t. 'x', we have

$$\frac{dy}{dx} = e^{-x}\frac{d}{dx}(x^3+2x^2+1) + (x^3+2x^2+1)\frac{d}{dx}(e^{-x})$$

$$= e^{-x}(3x^2+4x) + (x^3+2x^2+1)e^{-x}\times(-1)$$

$$= e^{-x}(3x^2+4x-x^3-2x^2-1)$$

$$= e^{-x}(-x^3+x^2+4x-1)$$

$$= -e^{-x}(x^3-x^2-4x+1)$$

Q.10 Find f'(x) if $f(x) = \lambda n\left(\sqrt{e^{2x}+e^{-2x}}\right)$

(Gujranwala Board 2014)(Multan Board 2013 G-II)
(Rawalpindi Board 2017 G-I)

Sol: Differentiating both sides w.r.t. 'x', we have

$$f'(x) = \frac{1}{\sqrt{e^{2x}+e^{-2x}}} \times \frac{d}{dx}\left(\sqrt{e^{2x}+e^{-2x}}\right)$$

$$= \frac{1}{\sqrt{e^{2x}+e^{-2x}}} \times \frac{1}{2}\left(e^{2x}+e^{-2x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(e^{2x}+e^{-2x}\right)$$

$$= \frac{1}{\sqrt{e^{2x}+e^{-2x}}} \times \frac{1}{2\sqrt{e^{2x}+e^{-2x}}} \times \left\{e^{2x}(2)+e^{-2x}(-2)\right\}$$

$$= \frac{1}{2\left(e^{2x}+e^{-2x}\right)} \times 2\left(e^{2x}-e^{-2x}\right) = \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$$

$$= \tanh 2x$$

Q.11 Find $\frac{dy}{dx}$ if $y = \ln\sqrt{\frac{x^2-1}{x^2+1}}$

(Sargodha Board 2017)(Sahiwal Board 2019)

Sol: $y = \ln\sqrt{\frac{x^2-1}{x^2+1}} = \ln\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}}$

$$= \frac{1}{2}\ln\left(\frac{x^2-1}{x^2+1}\right) = \frac{1}{2}(\ln(x^2-1) - \ln(x^2+1))$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x^2-1}\frac{d}{dx}(x^2-1) - \frac{1}{x^2+1}\frac{d}{dx}(x^2+1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{x^2-1}(2x) - \frac{1}{x^2+1}(2x)\right]$$

$$= \frac{2x}{2}\left[\frac{1}{x^2-1} - \frac{1}{x^2+1}\right]$$

$$= x\left[\frac{x^2+1-x^2+1}{(x^2-1)(x^2+1)}\right] = \frac{2x}{x^4-1}$$

Q.12 Find $\frac{dy}{dx}$ if $y = \lambda n(\tanh x)$

(Faisalabad Board 2013)(Lahore Board 2015 G-I)
(Multan Board 2018 G-II)(Gujranwala Board 2018)
(Faisalabad Board 2019 G-II)

Sol: Differentiating w.r.t. 'x' we have

$$\frac{dy}{dx} = \frac{1}{\tanh x}\times\frac{d}{dx}(\tanh x) = \frac{1}{\tanh x}\times\operatorname{sech}^2 x$$

$$= \frac{\cosh x}{\sinh x}\times\frac{1}{\cosh^2 x}$$

$$= \frac{1}{\sinh x\cosh x} = \frac{2}{2\sinh x\cosh x}$$

$$= \frac{2}{\sinh 2x} = 2\operatorname{cosech} 2x.$$

Q.13 Find $\frac{dy}{dx}$ if $y = \sinh^{-1}(x^3)$

(Rawalpindi Board 2017)(Multan Board 2017 G-I)
(Sahiwal, D.G. Khan Board 2014 G-II)

Sol: $y = \sinh^{-1} x^3$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+(x^3)^2}}\frac{d}{dx}x^3$$

$$= \frac{1}{\sqrt{1+x^6}}\cdot 3x^2$$

Q.14 Differentiate $(\lambda n\, x)^x$ w.r.t. 'x'

(Lahore Board 2014 G-II)(Multan Board 2018 G-I)
(Bahawalpur Board 2018)(Sahiwal Board 2019)

Sol. Let $y = (\lambda n\, x)^x \quad \Lambda\Lambda$ (i)

Taking λn with of both sides of (i), we have

$$\lambda n\, y = \lambda n\,[(\lambda n\, x)^x]$$

$$= x\,\lambda n\,(\lambda n\, x)$$

Differentiating both sides of (ii) w.r.t. 'x'

$$\frac{1}{y}\frac{dy}{dx} = 1\,.\,\lambda n\,(\lambda n\, x) + x\,.\,\frac{1}{\ln x}\cdot\frac{d}{dx}(\ln x)$$

$$= \lambda n\,(\lambda n\, x) + x\,.\,\frac{1}{\ln x}\cdot\frac{1}{x} = \ln(\ln x) + \frac{1}{\ln x}$$

$$\frac{dy}{dx} = y\left[\ln(\ln x) + \frac{1}{\ln x}\right] = (\ln x)^x\left[\ln(\ln x) + \frac{1}{\ln x}\right]$$

Q.15 Find $\frac{dy}{dx}$ if $y = e^{-2x} \sin 2x$

(Faisalabad Board 2019 G-II)(Gulranwala Board 2019 G-II)

Sol. $y = e^{-2x} \sin 2x$

Diff. w.r.t x

$$\frac{dy}{dx} = e^{-2x}\frac{d}{dx}\sin 2x + \frac{d}{dx}e^{-2x} - \sin 2x$$

$$= e^{-2x}\cos 2x\,(2) + e^{-2x}(-2)\sin 2x$$

$$= 2e^{-2x}[\cos 2x - \sin 2x]$$

Q.16 Find f ′ (x) if $f(x) = \sqrt{\ln (e^{2x} + e^{-2x})}$.

(Sahiwal Board 2019)(Lahore Board 2019 G-II)

Sol. $f(x) = \sqrt{\ln (e^{2x} + e^{-2x})}$

Diff. w.r.t x

$$f'(x) = \frac{1}{2}(\ln(e^{2x} + e^{-2x}))^{-1/2}\frac{d}{dx}\ln (e^{2x} + e^{-2x})$$

$$= \frac{1}{2}\frac{1}{\sqrt{\ln (e^{2x} + e^{-2x})}} \times \frac{1}{e^{2x} + e^{-2x}}\{e^{2x}(2) + e^{-2x}(-2)\}$$

$$= \frac{1}{2}\frac{1}{\sqrt{\ln (e^{2x} + e^{-2x})}} \times \frac{2\{e^{2x} - e^{-2x}\}}{e^{2x} + e^{-2x}}$$

$$= \frac{1}{\sqrt{\ln (e^{2x} + e^{-2x})}} \times \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$$

Q.17 Find $\frac{dy}{dx}$ if $y = (x + 1)^x$.

(Rawalpindi Board 2019)

Sol. See Long Question 1

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 $\frac{d}{dx}\log_a (2x)$ equals: *(Sahiwal Board 2014)*

(a) $\frac{1}{2x}\log_a e$ (b) $\frac{1}{x}\log_a e$

(c) $\frac{1}{x}\log_e a$ (d) $\frac{1}{x}\ln a$

Q.2 $\frac{d}{dx}(\ln 3x)$ equal: *(Sahiwal Board 2014)*

(a) $\frac{1}{3x}$ (b) $\frac{3}{x}$

(c) 3x (d) $\frac{1}{x}$

Q.3 $\frac{d}{dx}(2^x)$ equals: *(Sahiwal Board 2014)*

(a) $x\,2^{x-1}$ (b) $2^x \ln 2$

(c) $\ln 2^x$ (d) $x \ln 2$

Q.4 $\frac{d}{dx}(\sin h\,3x)$ equals: *(Sahiwal Board 2014)*

(a) $\cos h\,3x$ (b) $-\cos h\,3x$

(c) $3\cos h\,3x$ (d) $\cos h\,x$

Q.5 $\frac{d}{dx}\cos hx =$ *(D.G Khan Board 2017 G-I)*

(a) $-\sin hx$ (b) $\sin hx$

(c) $\sec hx$ (d) $\operatorname{cosec} hx$

Q.6 If $f(x) = \ln(x + 1)$, then $f'(x) =$ *(D.G.K Board 2014 G-I)*

(a) $x + 1$ (b) $\frac{1}{1 - x}$

(c) $\frac{1}{x + 1}$ (d) $1 - x$

Q.7 $\frac{d}{dx}\sin h\,3x =$ *(D.G.K Board 2014 G-I)*

(a) $3\sin h\,3x$ (b) $3\cos h\,3x$

(c) $\cos h\,3x$ (d) $\sin h\,3x$

Q.8 $\frac{d}{dx}a^x$ is equal to: *(Faisalabad Board 2016)*

(Multan Board 2013 G-II, Multan Board 2014 G-II)

(Sargodha Board 2017)(Multan Board 2017 G-I)

(D.G. Khan Board 2015 G-I)

(a) $x\,a^{x-1}$ (b) a^x

(c) $a^x \ln a$ (d) $\frac{a^x}{\ln a}$

Q.9 The derivative of $f(x) \equiv e^x$ equals: *(Multan Board 2014 G-I)*

(a) e^x (b) $x\,e^{x-1}$

(c) $\frac{e^{x-1}}{x - 1}$ (d) $\frac{e^{x+1}}{x + 1}$

Q.10 $\frac{d}{dx}\sin x^3$ is equal to: *(Multan Board 2014 G-I)*

(a) $\cos x^3$ (b) $-\cos x^3$

(c) $x^2 \sin x^3$ (d) $3x^2 \cos x^3$

Q.11 $\frac{d}{dx}\log_a x =$ ————

(Rawalpindi Board 2013, Sahiwal Board 2013, Bahawalpur Board 2014, Sargodha Board 2016, Lahore Board 2018 G-I)

(a) $\frac{\ln a}{x}$ (b) $\frac{x}{\ln a}$

(c) $\frac{1}{x}$ (d) $\frac{1}{x \ln a}$

Q.12 $\frac{d}{dx}(e^{\sqrt{x}}) =$ *(D.G Khan Board 2017 G-I)*

(a) $e^{\sqrt{x}}$ (b) $\frac{-e^{\sqrt{x}}}{\sqrt{x}}$

(c) $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$ (d) $\frac{\sqrt{x}\,e^{\sqrt{x}}}{2}$

Q.13 $\frac{d}{dx}\operatorname{cosec} hx =$ *(Bahawalpur Board 2016)*

(a) $\operatorname{cosec} hx \cot hx$ (b) $\sec hx \tan hx$

(c) $-\operatorname{cosec} hx \cot hx$ (d) $\operatorname{cosec} h^2 x$

Q.14 $\frac{d}{dx}(\ln x)$ is equal to: *(Rawalpindi Board 2014)*

(a) $\frac{1}{\ln x}$ (b) $\frac{1}{x}$

(c) x (d) $\ln x$

Q.15 $\frac{d}{dx}\ a^{\gamma x} =$ *(Rawalpindi Board 2017 G-II)*

(a) $\gamma a^x \ln a$ (b) $\gamma a^{\gamma x} \ln a$

(c) $\frac{a^{\gamma x}}{\ln a}$ (d) $\frac{a^{\gamma x}}{\gamma}$

Q.16 $\frac{d}{dx}[\lambda n (\text{Sin h } x)]$ equals *(Sargodha Board 2019)*

(a) Cot h x (b) tan h x

(c) − Cot h x (d) − tan h x

Q.17 $\frac{d}{dx}\tan h^{-1} x =$ *(Multan Board 2013 G-II)*

(a) $\frac{1}{1+x^2}$ (b) $\frac{-1}{1+x^2}$

(c) $\frac{1}{1-x^2}$ (d) $\frac{-1}{1-x^2}$

Q.18 If $y = e^{f(x)}$, then f '(x) will be equal to: *(Sargodha Board 2019)*

(a) $y\frac{dy}{dx}$ (b) $y\frac{dx}{dy}$

(c) $\frac{1}{y}\frac{dy}{dx}$ (d) $\frac{1}{y}\frac{dx}{dy}$

Q.19 $\frac{d}{dx}\left(2^{\sqrt{x}}\right)$ equals: *(Multan Board 2013 G-I)*

(a) $2^{\sqrt{x}}$ (b) $2^{\sqrt{x}} \ln 2$

(c) $\frac{2^{\sqrt{x}} \ln 2}{2\sqrt{x}}$ (d) $\frac{2^{\sqrt{x}}}{2\sqrt{x}}$

Q.20 If $y = \ln (\tan h\ x)$, then $\frac{dy}{dx}$ is: *(Multan Board 2013 G-I)*

(a) $\sec h^2 x \cot hx$ (b) $2 \sec hx$

(c) $\sec hx \cot h^2 x$ (d) $-2 \sec hx \cot hx$

Q.21 $\frac{d}{dx}(\ln (\ln x))$ *(Faisalabad Board 2013)*
(Gujranwala Board 2016)

(a) $\frac{1}{x}$ (b) $\frac{1}{x \ln a}$

(c) $\frac{1}{x \ln x}$ (d) $\frac{x}{\ln x}$

Q.22 If $y = \ln f(x)$, then $\frac{dy}{dx} =$ *(Faisalabad Board 2013)*

(a) $\frac{f'(x)}{f(x) \ln a}$ (b) $\frac{f'(x)}{f(x)}$

(c) f(x) (d) $\frac{1}{f(x)}$

Q.23 $\frac{d}{dx}\cos hx = ?$ *(D.G.K Board 2013 G-II)*
(Lahore Board 2018 G-II)(Faisalabad Board 2019)

(a) sin hx (b) −sin hx

(c) cosec hx (d) cos hx

Q.24 $\frac{d}{dx}\text{cosec hx} = ?$ *(D.G.K Board 2013 G-I)*

(a) sec hx tan hx (b) −cosec hx

(c) −cosec hx cot hx (d) $-\cot^2 hx$

Q.25 $\frac{d}{dx}(e^{f(x)}) = ?$ *(D.G.K Board 2013 G-I)*
(Lahore Board 2016 G-I)

(a) e (b) e^x

(c) $\frac{e^{f(x)}}{f(x)}$ (d) $e^{f(x)} f'(x)$

Q.26 $\frac{d}{dx} a^{f(x)} =$ *(D.G.K Board 2012 G-II)*
(Lahore Board 2013 G-II)

(a) $a^{f(x)}$ (b) $a^{f(x)} f'(x)$

(c) $a^{f(x)} \ln a$ (d) $a^{f(x)} f'(x) \ln a$

Q.27 $\frac{d}{dx}(a^x)$ equals: *(D.G.K Board 2011)*

(a) a^x (b) $a^x \ln a$

(c) 0 (d) $\frac{a^x}{\ln a}$

Q.28 The derivative of $e^{\ln x}$ w.r.t 'x' is equal to: *(D.G.K Board 2011)*

(a) 0 (b) −1

(c) $\frac{1}{x}$ (d) 1

Q.29 If $f(x) = \cos hx$, then $(f(x))^2 - (f'(x))^2 =$
(Multan Board 2017 G-I)(Lahore Board 2017 G-II)
(Faisalabad Board 2019 G-I)

(a) 1 (b) 0

(c) $\frac{1}{2}$ (d) 2^2

Q.30 $\frac{d}{dx}(\cos h^{-1} x)$ equals to: *(D.G.K Board 2010)*
(Faisalabad Board 2017)

(a) $-\frac{1}{\sqrt{x^2-1}}$ (b) $\frac{1}{\sqrt{1-x^2}}$

(c) $\frac{1}{\sqrt{x^2-1}}$ (d) $\frac{-1}{\sqrt{1-x^2}}$

Q.31 If $f(0) = e^{\sqrt{x}-1}$, then f'(x) equals to: *(D.G.K Board 2010)*

(a) $\frac{1}{2\sqrt{x}} e^{\sqrt{x}-1}$ (b) $\frac{-1}{2\sqrt{x}} e^{\sqrt{x}-1}$

(c) $\frac{1}{\sqrt{x}} e^{\sqrt{x}-1}$ (d) $\frac{-1}{\sqrt{x}} e^{\sqrt{x}-1}$

Q.32 If $y = \frac{e^x + 1}{e^x - 1}$, then $\frac{dy}{dx}$ is equal to:

(a) $-\frac{2e^x}{(e^x - 1)}$ (b) $\frac{e^x}{(e^x - 1)^2}$

(c) $-\frac{2\,e^x}{(e^x - 1)^2}$ (d) None of these

Q.33 $\frac{d}{dy}(\sin h^{-1}x) =$

(a) $\frac{1}{\sqrt{x^2 - 1}}$ (b) $-\frac{1}{\sqrt{x^2 - 1}}$

(c) $\frac{1}{\sqrt{1 + x^2}}$ (d) $-\frac{1}{\sqrt{1 + x^2}}$

Q.34 $\frac{d}{dy}(\log_a x) =$ *(D.G Khan Board 2017 G-I)*

(a) $\frac{1}{x}$ (b) $\frac{1}{x\, ln\, a}$

(c) $x.ln\, a$ (d) $x \cdot \frac{1}{ln\, a}$

Q.35 $\frac{d}{dx}(\tan h^{-1}x) =$

(Lahore Board 2016 G-I)(Sargodha Board 2017)

(a) $\frac{1}{1 - x^2}$ (b) $-\frac{1}{1 + x^2}$

(c) $\frac{1}{1 + x^2}$ (d) $-\frac{1}{1 - x^2}$

Q.36 $\frac{d}{dx}(\sin h\, x) =$

(a) $\sin h\, x$ (b) $\cos h\, x$

(c) $-\sin h\, x$ (d) $-\cos h\, x$

Q.37 $\frac{d}{dx}(\tan h\, x) =$

(a) $\text{sech}^2\, x$ (b) $\cos h\, x$

(c) $-\sin h\, x$ (d) $-\cos h\, x$

Q.38 $\frac{d}{dx}(\text{cosec } h\, x) =$

(a) $\text{cosec } h\, x \cot h\, x$ (b) $\text{cosec } h^2 x$

(c) $-\text{cosec } h\, x \cot h\, x$ (d) $-\text{cosec } h^2 x$

Q.39 $\frac{d}{dx}(\sec h\, x) =$ *(Rawalpindi Board 2017 G-I)*

(Lahore Board 2018 G-II)

(a) $\sec h\, x \tan h\, x$ (b) $-\sec h^2 x$

(c) $-\sec h\, x \tan h\, x$ (d) $\text{sech}^2 x$

Q.40 $\frac{d}{dx}(\cot h\, x) =$ *(Multan Board 2019 G-I)*

(a) $\text{cosec } h\, x \cot h\, x$ (b) $-\text{cosec } h^2 x$

(c) $-\text{cosec } h\, x \cot h\, x$ (d) $\text{cosec } h^2 x$

Q.41 $\frac{d}{dx}(\cot h^{-1}\, x) =$ (Multan Board 19 G-I)

(a) $\frac{1}{1 - x^2}$ (b) $-\frac{1}{1 + x^2}$

(c) $\frac{1}{1 + x^2}$ (d) $-\frac{1}{1 - x^2}$

Q.42 $\frac{d}{dx}(\cos h^{-1}\, x) =$ *(Gujranwala Board 2016)*

(a) $-\frac{1}{\sqrt{x^2 - 1}}$ (b) $\frac{1}{\sqrt{x^2 - 1}}$

(c) $\frac{1}{\sqrt{1 + x^2}}$ (d) $-\frac{1}{\sqrt{1 + x^2}}$

Q.43 $\frac{d}{dx}(\text{cosec } h^{-1}\, x) =$

(a) $\frac{1}{|x|\sqrt{1 + x^2}}$ (b) $-\frac{1}{|x|\sqrt{1 + x^2}}$

(c) $\frac{1}{|x|\sqrt{x^2 - 1}}$ (d) $-\frac{1}{|x|\sqrt{x^2 - 1}}$

Q.44 $\frac{d}{dx}(\sec h^{-1}\, x) =$

(a) $\frac{1}{|x|\sqrt{1 - x^2}}$ (b) $\frac{1}{|x|\sqrt{1 - x^2}}$

(c) $-\frac{1}{|x|\sqrt{x^2 - 1}}$ (d) $\frac{1}{|x|\sqrt{x^2 - 1}}$

Q.45 $\frac{d}{dx}(\log |x|) =$

(a) 1 (b) -1

(c) $\frac{1}{x}$ (d) $-\frac{1}{x}$

Q.46 $\frac{d}{dx}(\log \tan x) =$

(a) $\text{cosec } 2x$ (b) $2\,\text{cosec } 2x$

(c) $\text{cosec}\left(\frac{1}{x}\right)$ (d) $\text{cosec } (-2x)$

Q.47 If $y = x^x$, $\frac{dy}{dx} =$

(a) $x^x (1 + \log x)$ (b) $x^x\left(1 + \frac{1}{x}\right)$

(c) $(1 + \log x)$ (d) None of these

Q.48 $\frac{d}{dx}\{\log (\sec x + \tan x)\} =$

(a) $\cos x$ (b) $\sec x$

(c) $\tan x$ (d) $\cot x$

Q.49 $\frac{d}{dx}(xe^{x^2}) =$

(a) $2x^2 e^{x^2} + e^{x^2}$ (b) $x^2 e^{x^2} + e^{x^2}$

(c) $e^x 2x^2 + e^{x^2}$ (d) None of these

Q.50 $\frac{d}{dx}(e^{x^3}) =$

(a) $3x\,e^{x^3}$ (b) $3x^2\,e^{x^3}$

(c) $3x\,(e^{x^3})^2$ (d) $2x^2\,e^{x^3}$

Q.51 If $y = \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}$, $\frac{dy}{dx} =$

(a) $\frac{-8}{(e^{2x}-e^{-2x})^2}$ (b) $\frac{8}{(e^{2x}-e^{-2x})^2}$

(c) $\frac{-4}{(e^{2x}-e^{-2x})^2}$ (d) $\frac{4}{(e^{2x}-e^{-2x})^2}$

Q.52 $\frac{d}{dx} ln\ 2x =$ ——— *(Gujranwala Board 2014)*

(a) $\frac{1}{2x}$ (b) $2x$

(c) $\frac{2}{x}$ (d) $\frac{1}{x}$

Q.53 If $y = ln\ x$, then $\frac{dy}{dx} =$ ———

(Lahore Board 2014, 2015 G-II) (Bahawalpur Board 2016)

(a) $-\frac{1}{x}$ (b) x^{-1}

(c) x (d) $-2x$

Q.54 $\frac{d}{dx}(\tan x)$ is equal to: *(Lahore Board 2014 G-II)*

(a) $ln \cos x$ (b) $-ln \cos x$

(c) $-sec^2 x$ (d) $sec^2 x$

Q.55 If $y = ln\ (\sin x)$, then $\frac{dy}{dx}$ equals:

(Lahore Board 2014, 2016 G-I)

(a) $\tan x$ (b) $\cot x$

(c) $-\tan x$ (d) $-\cot x$

Q.56 If $y = \sin h^{-1}x$, then $\frac{dy}{dx} =$

(Lahore Board 2013, 2015 G-II, 2015 G-I)

(a) $\frac{1}{\sqrt{x^2+1}}$ (b) $\frac{1}{\sqrt{x^2-1}}$

(c) $\frac{1}{x^2+1}$ (d) $\frac{1}{x^2-1}$

Q.57 $\frac{d}{dx}$ Sin h2x = ———

(Lahore Board 2012 G-II)(Gujranwala Board 2019)

(a) $2 \cos h2x$ (b) $2 \sin h2x$

(c) $-2 \cos h\ 2x$ (d) $-2 \sin h\ 2x$

Q.58 $\frac{d}{dx}(3^{3x}) =$ *(Lahore Board 2012 G-I)*

(a) $3^{3x}\ ln\ 3$ (b) $3^{3x}\ ln\ 9$

(c) $3^{3x}\ ln\ 27$ (d) $3^{3x}\ ln\ 18$

Q.59 $\frac{d}{dx}(e^{\cos x})$ equals: *(Lahore Board 2015 G-II)*

(a) $-\sin x\ e^{\cos x}$ (b) $\sin x\ e^{\cos x}$

(c) $\cos x\ e^{\sin x}$ (d) $-\cos x\ e^{\sin x}$

Q.60 If $f(x) = e^{ax}$, then $f'(x)$ is equal to:

(Lahore Board 2015 G-I)

(a) $\frac{e^{ax}}{a}$ (b) $-\frac{e^{ax}}{a}$

(c) $a\,e^{ax}$ (d) $-a\,e^{ax}$

Q.61 $\frac{d}{dx}(e^{ln\,x^2}) =$ *(Faisalabad Board 2016)*

(a) $e^{ln\,x^2}$ (b) $e^{2x\,ln\,x^2}$

(c) $2x\,e^{ln\,x^2}$ (d) $2x$

Q.62 $\frac{d}{dx}(\ln x^m)^k$ *(Lahore Board 2017 G-II)*

(a) $\frac{mk}{x}(\ln x^m)^{k-1}$ (b) $\frac{k}{x^m}(\ln x)^{k-1}$

(c) $\frac{1}{x^{mk}}$ (d) $\frac{mk}{x}$

Q.63 $\frac{d}{dx}(ln\ e^x) =$ *(Bahawalpur Board 2019)*

(a) e^x (b) $\frac{1}{e^x}$

(c) 1 (d) e^{x-1}

Q.64 $\frac{d}{dx} ln\ (f(x)) =$ *(Sargodha Board 2016)*

(a) $\frac{1}{x}$ (b) $\frac{1}{f(x)}$

(c) $\frac{f'(x)}{f(x)}$ (d) $f(x)\,.\,f'(x)$

Q.65 $\frac{d}{dx}(e^{x^2+1}) =$ *(Sargodha Board 2016)*

(a) e^{x^2+1} (b) $2x\,e^{x^2+1}$

(c) $2e^{x^2+1}$ (d) $-2x\,e^{x^2+1}$

Q.66 $\frac{d}{dx} \ln\left(\frac{1}{x}\right) =$ *(D.G Khan Board 2017 G-I)*

(a) x (b) $-x$

(c) $\frac{1}{x}$ (d) $-\frac{1}{x}$

Q.67 $\frac{d}{dx}(\ln(e^x + e^{-x})) =$ *(Faisalabad Board 2017)*

(a) $\frac{e^x+e^{-x}}{e^x-e^{-x}}$ (b) $\frac{e^x-e^{-x}}{e^x+e^{-x}}$

(c) $\frac{e^x-e^{-x}}{-e^x+e^{-x}}$ (d) $\frac{-e^x+e^{-x}}{e^x-e^{-x}}$

Q.68 $\frac{d}{dx} e^{x+h} =$ *(Lahore Board 2017 G-I)*

(a) $\frac{e^{x+4}}{\ln h}$ (b) $\frac{e^{x+4}}{\ln x}$

(c) e^{x+4} (d) he^{x+4}

Q.69 If $y = \cosh x$, then $\frac{dy}{dx} =$:

(Faisalabad Board 2019)

(a) $-\sinh y$ (b) $\sinh y$

(c) $-\cosh x$ (d) $\sinh x$

Q.70 $\frac{d}{dx}(\ln \cos x) =$: *(Faisalabad Board 2019)*

(a) $\tan x$ (b) $\cot x$

(c) $-\tan x$ (d) $-\cot x$

Q.71 $\frac{d}{dx}(\tan^{-1} 3x) =$: *(Faisalabad Board 2019 G-II)*

(a) $\frac{1}{1+3x}$ (b) $\frac{3}{1+3x}$

(c) $\frac{1}{1+9x^2}$ (d) $\frac{3}{1+9x^2}$

Q.72 $\frac{d}{dx}(e^{\sin x}) =$: *(Faisalabad Board 2019 G-II)*

(a) $\cos x$ (b) $e^{\sin x} \cos x$

(c) $e^{\sin x} \sin x$ (d) $\sin x$

Q.73 $\frac{d}{dx}(\ln 2x) =$ *(Gujranwala Board 2019 G-II)*

(a) $\frac{1}{2x}$ (b) $\frac{1}{x}$

(c) $-\frac{1}{2x}$ (d) $2x$

Q.74 $\frac{d}{dx}(e^{x^2})$ *(Lahore Board 2019)*

(a) e^{x^2} (b) $2e^{x^2}$

(c) $2xe^{x^2}$ (d) $2e$

EXERCISE 2.7

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find y_2 if $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

(D.G Khan Board 2017 G-I)
(Gujranwala Board 2012, 2013)
(Lahore Board 2012 G-II) (Bahawalpur Board 2019)

Sol: $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

Differentiate w.r.t x

$y_1 = 2(5x^4) - 3(4x^3) + 4(3x^2) + 1$

$= 10x^4 - 12x^3 + 12x^2 + 1$

Differentiate w.r.t x

$y_2 = 10(4x^3) - 12(3x^2) + 12(2x)$

$= 40x^3 - 36x^2 + 24x$

Q.2 Find y_4 if $y = \ln(x^2 - 9)$

(D.G Khan Board 2014 G-II)
(Lahore Board 2014 G-I, 2014 G-II)

Sol: $y = \ln(x^2 - 9)$

$y = \ln\{(x-3)(x+3)\}$

$y = \ln(x-3) + \ln(x+3)$

Diff.

$y_1 = \frac{1}{x-3} + \frac{1}{x+3}$

Diff.

$y_2 = \frac{-1}{(x-3)^2} + \frac{-1}{(x+3)^2}$

Diff.

$y_3 = \frac{(-1)(-2)}{(x-3)^3} + \frac{(-1)(-2)}{(x+3)^3}$

Diff.

$y_4 = \frac{(-1)(-2)(-3)}{(x-3)^4} + \frac{(-1)(-2)(-3)}{(x+3)^4}$

$= -6\left(\frac{1}{(x-3)^4} + \frac{1}{(x+3)^4}\right)$

Q.3 Find y_4 if $y = \sin 3x$

(Lahore Board 2011 G-II, 2016 G-I)

Sol: Differentiating w.r.t. 'x', we have

$y_1 = (\cos 3x) \times 3 = 3\cos 3x$

Differentiating again w.r.t. 'x', we have

$y_2 = 3(-\sin 3x) \times 3 = -9 \sin 3x$

Differentiating again w.r.t. 'x', we have

$y_3 = -9\frac{d}{dx}(\sin 3x) = -9 \times (\cos 3x) \times 3$

$= -27 \cos 3x$

Differentiating again w.r.t. 'x', we have

$y_4 = -27\frac{d}{dx}(\cos 3x) = -27 \times (-\sin 3x) \times 3$

$= 81 \sin 3x$

Q.4 Find y_2 if $y = x^2 \cdot e^{-x}$

(Sahiwal Board 2014 G-II) (Rawalpindi Board 2017)
(Gujranwala Board 2010, 2017)

Sol: Differentiating w.r.t. 'x', we have

$y_1 = e^{-x}\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(e^{-x})$

$y_1 = e^{-x} \times 2x + x^2 e^{-x} \times (-1)$

$y_1 = e^{-x}(2x - x^2)$

Again differentiating w.r.t. 'x', we have

$y_2 = e^{-x}\frac{d}{dx}(2x - x^2) + (2x - x^2)\frac{d}{dx}(e^{-x})$

$= e^{-x}(2 - 2x) + (2x - x^2)e^{-x}(-1)$

$= e^{-x}(2 - 2x - 2x + x^2) = e^{-x}(x^2 - 4x + 2)$

Q.5 Find y_2 if $x^3 - y^3 = a^3$.
(Lahore Board 2013 G-I)(Multan Board 2013 G-I)

Ans. Given that

$x^3 - y^3 = a^3$

$\Rightarrow y^3 = x^3 - a^3$

Diff. w.r.t 'x'

$$\frac{d}{dx}(y^3) = \frac{d}{dx}(x^3 - a^3)$$

$$3y^2\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{3y^2} = \frac{x^2}{y^2}$$

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

Diff. .w.r.t 'x'

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{x^2}{y^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{y^2\frac{d}{dx}x^2 - x^2\frac{d}{dx}(y^2)}{(y^2)^2} = \frac{y^2.2x - x^2.2y\frac{dy}{dx}}{y^4}$$

$$= \frac{1}{y^4}\left\{2xy^2 - 2yx^2\frac{dy}{dx}\right\} = \frac{d^2y}{dx^2} = \frac{2yx}{y^4}\left\{y - x\frac{dy}{dx}\right\}$$

$$= \frac{2x}{y^3}\left\{y - x\left(\frac{x^2}{y^2}\right)\right\} \quad \Theta\ \frac{dy}{dx} = \frac{x^2}{y^2}$$

$$= \frac{2x}{y^3}\left\{y - \frac{x^3}{y^2}\right\} = \frac{2x}{y^3}\left\{\frac{y^3 - x^3}{y^2}\right\} = \frac{-2x(x^3 - y^3)}{y^5}$$

$$= \frac{-2xa^3}{y^5} \quad \Theta\ x^3 - y^3 = a^3$$

$$y^2 = \frac{-2a^3x}{y^5} \quad \text{Ans.}$$

Q.7 Find $\frac{dy}{dx}$ when $y = a\cos(\lambda n\, x) + b\sin(\lambda n\, x)$
(Lahore Board 2015 G-I)

Sol: See Long Question 5

Q.8 If $x = a(\theta + \sin)$ and $y = (1 + \cos\theta)$, show that : $y^2\frac{d^2y}{dx^2} + a = 0$.
(Rawalpindi Board 2013)
(Multan Board 2017 G-I)

Sol: $x = a(\theta + \sin\theta)$

$$\Rightarrow \frac{dx}{d\theta} = \frac{d}{d\theta}[a(\theta + \sin\theta)]$$

$$\frac{dx}{d\theta} = a.\frac{d}{d\theta}[\theta + \sin\theta]$$

$$\Rightarrow \frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$\frac{d\theta}{dx} = \frac{1}{a(1 + \cos\theta)}$$

and $y = a(1 + \cos\theta)$

Differentiating w.r.t "x", we have

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}[a(1 + \cos\theta)]$$

$$\frac{dy}{dx} = a.\frac{d}{dx}[(1 + \cos\theta)]$$

$$= a.(-\sin\theta).\frac{d\theta}{dx}$$

$$= -a\sin\theta.\frac{1}{a(1 + \cos\theta)}$$

$$= -\frac{\sin\theta}{1 + \cos\theta}$$

Differentiating w.r.t "x", we have

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left[\frac{\sin\theta}{1 + \cos\theta}\right]$$

$$= -\frac{d\theta}{dx}.\frac{d}{d\theta}\left[\frac{\sin\theta}{1 + \cos\theta}\right]$$

$$= -\frac{d\theta}{dx}.\frac{\cos\theta.(1 + \cos\theta) - \sin\theta(-\sin\theta)}{(1 + \cos\theta)^2}$$

$$= -\frac{1}{a(1 + \cos\theta)}.\frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1 + \cos\theta)^2}$$

$$= -\frac{\cos\theta + 1}{a(1 + \cos\theta)^3} = \frac{-1}{a(1 + \cos\theta)^2}$$

But $1 + \cos\theta = \frac{y}{a}$, so

$$\frac{d^2y}{dx^2} = \frac{-1}{a\left(\frac{y}{a}\right)^2} = \frac{-1}{a.\frac{y^2}{a^2}} = \frac{-1}{\frac{y^2}{a}} = \frac{-a}{y^2}$$

$$y^2.\frac{d^2y}{dx^2} = -a \quad \Rightarrow \quad y^2\frac{d^2y}{dx^2} + a = 0$$

Q.9 Find y_1, if $x^3 - y^3 = a^3$ ***(Lahore Board 2013)***

Ans. Diff. w.r.t. 'x'

$$3x^2 - 3y^2\frac{dy}{dx} = 0$$

$$x^2 - y^2\frac{dy}{dx} = 0$$

$$y^2\frac{dy}{dx} = x^2$$

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

LONG QUESTIONS

Q.1 If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} + 2y = 0$

(Rawalpindi Board 2014)(Lahore Board 2019 G-II)

Sol: $y = e^x \sin x$

Differentiating w.r.t "x", we have

$$\frac{dy}{dx} = e^x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} e^x$$

$$\frac{dy}{dx} = e^x \cos x + \sin x\, e^x = e^x (\cos x + \sin x)$$

Differentiating w.r.t "x" again, we have

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx} (\cos x + \sin x) + (\cos x + \sin x) \frac{d}{dx} e^x$$

$$\frac{d^2y}{dx^2} = e^x (-\sin x + \cos x) + (\cos x + \sin x) e^x$$

$$\frac{d^2y}{dx^2} = e^x (-\sin x + \cos x + \cos x + \sin x)$$

$$\frac{d^2y}{dx^2} = e^x (2 \cos x)$$

$$\text{L.H.S} = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y$$

$$= e^x (2 \cos x) - 2e^x (\cos x + \sin x) + 2 (e^x \sin x)$$
$$= 2e^x (\cos x - \cos x - \sin x + \sin x) = 2e^x (0) = 0$$
$$= \text{R.H.S}$$

Q.2 If $y = e^{ax} \sin bx$, show that

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2) y = 0$$

(Multan Board 2018 G-II)

Sol: $y = e^{ax} \sin bx$

Differentiating w.r.t "x", we have

$$\frac{dy}{dx} = e^{ax} \frac{d}{dx} \sin bx + \sin bx \frac{d}{dx} e^{ax}$$

$$\frac{dy}{dx} = e^{ax} \cos bx.b + \sin bx\, e^{ax}.a$$

$$\frac{dy}{dx} = e^{ax} (b \cos bx + a \sin bx)$$

$$\frac{d^2y}{dx^2} = e^{ax} \frac{d}{dx} (b \cos bx + a \sin bx)$$
$$+ (b \cos bx + a \sin bx) \frac{d}{dx} e^{ax}$$

$$\frac{d^2y}{dx^2} = e^{ax} (-b \sin bx.b + a \cos bx.b)$$
$$+ (b \cos bx + a \sin bx) e^{ax} a$$

$$\frac{d^2y}{dx^2} = e^{ax} (-b^2 \sin bx + ab \cos bx$$
$$+ ab \cos bx + a^2 \sin bx)$$

$$\frac{d^2y}{dx^2} = e^{ax} (-b^2 \sin bx + 2ab \cos bx + a^2 \sin bx)$$

$$\text{L.H.S} = \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2) y$$

$$= e^{ax} (-b^2 \sin bx + 2ab \cos b\, x + a^2 \sin bx)$$
$$-2ae^{ax} (b \cos bx + a \sin bx) + (a^2 + b^2) e^{ax} \sin bx$$
$$= e^{ax} (-b^2 \sin bx + 2ab\ \cos bx + a^2 \sin bx - 2ab \cos bx$$
$$- 2a^2 \sin bx + a^2 \sin bx + b^2 \sin bx)$$
$$= a^{ax} (0) = 0 = \text{R.H.S}$$

Q.3 If $y = (\cos^{-1} x)^2$, prove that $(1 - x^2)y_2 - xy_1 - 2 = 0$

(Multan Board 2012, 2015 G-I)(Lahore Board 2012 G-I)
(Sahiwal Board 2018)(Faisalabad Board 2019 G-I)

Sol: $y = (\cos^{-1} x)^2$

Differentiating w.r.t "x", we have

$$\frac{dy}{dx} = 2(\cos^{-1} x)^{2-1} \times \frac{d}{dx} (\cos^{-1} x)$$

$$y_1 = 2 (\cos^{-1} x) \times \frac{-1}{\sqrt{1 - x^2}}$$

Multiplying by $\sqrt{1 - x^2}$

$$\sqrt{1 - x^2}\, y_1 = -2 (\cos^{-1} x)$$

Differentiating w.r.t "x" again, we have

$$\sqrt{1 - x^2} \frac{d}{dx} y_1 + y_1 \frac{d}{dx} \sqrt{1 - x^2} = -2 \frac{d}{dx} (\cos^{-1} x)$$

$$\sqrt{1 - x^2}\, y_2 + y_1 \frac{1}{2\sqrt{1 - x^2}} (-2x) = -2 \frac{-1}{\sqrt{1 - x^2}}$$

$$\sqrt{1 - x^2}\, y_2 - \frac{xy_1}{\sqrt{1 - x^2}} = \frac{2}{\sqrt{1 - x^2}}$$

$$\frac{(1 - x^2) y_2 - xy_1}{\sqrt{1 - x^2}} = \frac{2}{\sqrt{1 - x^2}}$$

Multiplying by $\sqrt{1 - x^2}$

$$(1 - x^2)\, y_2 - xy_1 = 2$$
$$(1 - x^2)\, y_2 - xy_1 - 2 = 0 \quad \text{Hence proved}$$

Q.4 If $y = a \cos (ln\, x) + b \sin (ln\, x)$, Prove that

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$$

(Gujranwala Board 2013, 2014)
(Rawalpindi Board 2013)(Bahawalpur Board 2014)
(Multan Board 2016 G-II)(Sargodha Board 2018)

Sol: $y = a \cos (\ln x) + b \sin (\ln x)$

Differentiating w.r.t "x", we have

$$\frac{dy}{dx} = a \frac{d}{dx} \cos (\ln x) + b \frac{d}{dx} \sin (\ln x)$$

$$\frac{dy}{dx} = a(-\sin (\ln x)) \frac{1}{x} + b(\cos (\ln x)) \frac{1}{x}$$

$$\frac{dy}{dx} = -a \sin (\ln x) \frac{1}{x} + b \cos (\ln x) \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x} [-a \sin (\ln x) + b \cos (\ln x)]$$

$$x \frac{dy}{dx} = -a \sin (\ln x) + b \cos (\ln x)$$

Differentiating w.r.t "x" again, we have

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 1 = -a\cos(\ln x)\frac{1}{x} - b\sin(\ln x)\frac{1}{x}$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}[a\cos(\ln x) + b\sin(\ln x)]$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}[y]$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \quad \text{Hence proved}$$

Q.5 If $x = a(\theta - \sin\theta)$; $y = a(1 + \cos\theta)$ then prove that $y^2\frac{d^2y}{dx^2} + a = 0$ *(Faisalabad Board 2019)*

Sol: $x = a(\theta + \sin\theta)$

$$\Rightarrow \frac{dx}{d\theta} = \frac{d}{d\theta}[a(\theta + \sin\theta)]$$

$$\frac{dx}{d\theta} = a\cdot\frac{d}{d\theta}[\theta + \sin\theta]$$

$$\Rightarrow \frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$\frac{d\theta}{dx} = \frac{1}{a(1+\cos\theta)}$$

and $y = a(1 + \cos\theta)$

Differentiating w.r.t "x", we have

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}[a(1 + \cos\theta)]$$

$$\frac{dy}{dx} = a\cdot\frac{d}{dx}[(1 + \cos\theta)] = a\cdot(-\sin\theta)\cdot\frac{d\theta}{dx}$$

$$= -a\sin\theta\cdot\frac{1}{a(1+\cos\theta)} = -\frac{\sin\theta}{1+\cos\theta}$$

Differentiating w.r.t "x", we have

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left[\frac{\sin\theta}{1+\cos\theta}\right] = -\frac{d\theta}{dx}\cdot\frac{d}{d\theta}\left[\frac{\sin\theta}{1+\cos\theta}\right]$$

$$= -\frac{d\theta}{dx}\cdot\frac{\cos\theta\,.\,(1+\cos\theta) - \sin\theta(-\sin\theta)}{(1+\cos\theta)^2}$$

$$= -\frac{1}{a(1+\cos\theta)}\cdot\frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1+\cos\theta)^2}$$

$$= -\frac{\cos\theta + 1}{a(1+\cos\theta)^3} = \frac{-1}{a(1+\cos\theta)^2}$$

But $1 + \cos\theta = \frac{y}{a}$, so

$$\frac{d^2y}{dx^2} = \frac{-1}{a\left(\frac{y}{a}\right)^2} = \frac{-1}{a\cdot\frac{y^2}{a^2}} = \frac{-1}{\frac{y^2}{a}} = \frac{-a}{y^2}$$

$$y^2\cdot\frac{d^2y}{dx^2} = -a \quad \Rightarrow \quad y^2\frac{d^2y}{dx^2} + a = 0$$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 If $f(x) = -\sin x$, then $f'''(\cos^{-1}x) =$ *(D.G.K Board 2015 G-II)*

(a) $\cos x$ (b) $-x$
(c) $-\sin x$ (d) x

Q.2 If $y = e^x$, then $y_3 =$ *(D.G.K Board 2014 G-II)*

(a) e^x (b) e^{2x}
(c) e^{3x} (d) e^{-x}

Q.3 If $y = u\,v$, where u, v are functions of x, then y_2 is:

(a) $u\,v_2 + 2u_1v_1 + u\,v_2$ (b) $u\,v_2 + u_1v_1 + u_2v$
(c) $u\,v_2 + u_1v_1 + u_2v$ (d) $u\,v_2 + 2u_1v_2 + u_2v$

Q.4 If $y = e^{3x}$, then y_3 is: *(Bahawalpur Board 2014)*

(a) e^{3x} (b) e^3
(c) $9e^{3x}$ (d) $27\,e^{3x}$

Q.5 If $y = e^{ax}$, then $y_4 =$ *(D.G.K Board 2015 G-I)*

(a) a^4e^{ax} (b) $2\frac{e^{ax}}{a}$
(c) $3e^{ax}$ (d) $x\,e^{ax}$

Q.6 If $y = e^{2x}$, then y_4 is equal to: *(D.G.K Board 2010)* *(Bahawalpur Board 2018)*

(a) e^{2x} (b) $2e^{2x}$
(c) $4e^{2x}$ (d) $16e^{2x}$

Q.7 If $y = \sin 3x$, then $y_2 =$ ———— *(Lahore Board 2014 G-II)(Rawalpindi Board 2018)* *(Gujranwala Board 2019 G-II)*

(a) $3\cos 3x$ (b) $9\cos 3x$
(c) $-9\sin 3x$ (d) $9\sin 3x$

Q.8 If $y = e^{ax}$, then $\frac{d^2y}{dx^2} =$ ———— *(Lahore Board 2012 G-II)*

(a) $-a^2e^{-ax}$ (b) $a\,e^{-ax}$
(c) $-ae^{-ax}$ (d) a^2e^{-ax}

Q.9 If $f(x) = \cos x$, then $f''(0)$ is equal to. *(D.G.K Board 2010)*

(a) -1 (b) 0
(c) 1 (d) 2

Q.10 If $y = \cos(ax + b)$, then y_4 equals to. *(D.G.K Board 2010)*

(a) $-a^4\sin(ax + b)$ (b) $a^4\sin(ax + b)$
(c) $a^4\cos(ax + b)$ (d) $-a^4\cos(ax + b)$

Q.11 If $y = \cos x$, then. *(Lahore Board 2012 G-I)*

(a) $y_4 + y = 0$ (b) $y_4 - y = 0$
(c) $y_2 - y = 4$ (d) $y_3 - y = 0$

Q.12 If $f(x) = \sin x$, then $f''(0) =$

(a) 1 (b) -1
(c) 0 (d) x

Q.13 $\frac{d^2}{dx^2}\cos 3x =$ *(Multan Board 2018 G-II)*

(a) 3 cosh 3x (b) 3 sinh 3x

(c) −9 cosh 3x (d) 9 cosh 3x

Q.14 If $y = \lambda nx$ then $y_2 =$:

(Faisalabad Board 2019 G-II)

(a) $\frac{1}{x}$ (b) $\frac{-1}{x}$

(c) $\frac{-1}{x^2}$ (d) $\frac{1}{x^2}$

Q.15 If $f(x) = \sin x$, then $f''\left(\frac{\pi}{2}\right) =$:

(Multan Board 2019)

(a) 0 (b) 1

(c) 2 (d) −1

EXERCISE 2.8

SHORT ANSWERS TO THE QUESTIONS

Q.1 Write Maclaurin's series expansion.

(Faisalabad Board 2013)(Sargodha Board 2017) (Lahore Board 2019 G-II)

Ans. The maclaurins series expansion of a function f(x) is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$$

Q.2 Prove that $e^x = 1 + x + \frac{x^2}{|2} = \frac{x^3}{|3} + \_\_\_\_\_$

(Multan Board 2013)(Multan Board 2014, 2015 G-II) (Sargodha Board 2016)

Ans. $e^x = 1 + x + \frac{x^2}{|2} = \frac{x^3}{|3} + \_\_\_\_\_$

Proof: By using Maclaurin series expression.

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots$

Let $f(x) = e^x$ Now $f(0) = e^0 = 1$

$f'(x) = e^x$ Now $f'(0) = e^0 = 1$

$f'(x) = e^x$ Now $f''(0) = e^0 = 1$

$f''(x) = e^x$ Now $f'''(0) = e^0 = 1$

Putting all these vales in (1)

$$e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(1) + \ldots\ldots$$

$$\Rightarrow e^x = 1 + x + \frac{x^2}{|2} + \frac{x^3}{|3} + \frac{x^4}{|4} + \ldots\ldots$$

Hence proved

Q.3 By Maclaurin's series prove that

$$e^{2x} = 1 + 2x + \frac{4x^2}{|2} + \frac{8x^3}{|3} + \ldots$$

(Lahore Board 2012 G-II)(Faisalabad Board 2019)

Sol: Let $f(x) = e^{2x}$

$f'(x) = e^{2x}(2)$

$f''(x) = 2^2 e^{2x}$

$f'''(x) = 2^3 e^{2x}$

and so on …

Putting x = 0, we have

$f(0) = e^{2(0)} = 1$

$f'(0) = 2e^{2(0)} = 2$

$f''(0) = 2^2 e^{2(0)} = 4$

$f'''(0) = 2^3 e^{2(0)} = 8$

By using Maclaurin series expansion, we have

$$f(x) = f(0) + \frac{f'(0)}{|1}x + \frac{f''(0)}{|2}x^2 + \frac{f'''(0)}{|3}x^3 + \ldots$$

$$e^{2x} = 1 + \frac{2}{|1}x + \frac{4}{|2}x^2 + \frac{8}{|3}x^3 + \ldots$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{|2} + \frac{8x^3}{|3} + \ldots$$

Q.4 Expand a^x in the Maclaurin series.

(Rawalpindi Board 2016)(Bahawalpur Board 2018)

Sol. Let $f(x) = a^x$. Then

$f'(x) = a^x \lambda n\, a$,

$f''(x) = a^x (\lambda n\, a)^2$, $f'''(x) a^x (\lambda n\, a)^3$,

$f(4)(x) = a^x (\lambda n\, a)^4$, Λ, $f^{(n)}(x) = a^x (\lambda n\, a)^n$.

Putting x = 0 in f(x), f '(x), f ''(x), f '''(x), $f^{(4)}(x)$, Λ $f^{(n)}(x)$, we get

$f(0) = a^0 = 1$, $f'(0) = a^0 \lambda n\, a = \lambda n\, a$, $f''(0) = (\lambda n\, a)^2$, $f'''(0) = (\lambda n\, a)^3$,

$f^{(4)}(0) = (\lambda n\, a)^4$, Λ, $f^{(n)}(0) = (\lambda n\, a)^n$.

Substituting these values in the formula

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{|2}x^2 + \frac{f'''(0)}{|3}x^3 + \Lambda + \frac{f(n)(0)}{|n}x^n + \Lambda,$$

we have

$$a^x = 1 + (\lambda n\, a)x + \frac{(\ln a)^2}{|2}x^2 + \frac{(\ln a)^3}{|3}x^3 + \Lambda + \frac{(\ln a)^n}{|n}x^n + \Lambda$$

Q.5 Find the Maclaurin series for sin x

Sol. Let $f(x) = \sin x$, Then $f(0) = \sin 0 = 0$.

$f'(x) = \cos x$ and $f'(0) = \cos 0 = 1$;

$f''(x) = -\sin x$ and $f''(0) = -\sin 0 = 0$;

$f'''(x) = -\cos x$ and $f'''(0) = -\cos 0 = -1$;

$f^{(4)}(x) = -(-\sin x) = \sin x$ and $f(4)(0) = \sin 0 = 0$.

$f^{(5)}(x) = \cos x$ and $f(5)(0) = \cos 0 = 1$,

$f^{(6)}(x) = -\sin x$ and $f(6)(0) = 0$;

$f^{(7)} = -\cos x$ and $f(7)(0) = -1$.

Putting these values in the formula

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{\lfloor 2}x^2 + \frac{f'''(0)}{\lfloor 3}x^3 + \frac{f^{(4)}(0)}{\lfloor 4}x^4 + \frac{f^{(5)}(0)}{\lfloor 5}x^5 + \cdots$$, we have

$$\sin x = 0 + 1\cdot x + \frac{0}{\lfloor 2}x^2 + \frac{-1}{\lfloor 3}x^3 + \frac{0}{\lfloor 4}x^4 + \frac{1}{\lfloor 5}x^5 + \cdots$$

$$= x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots$$

Q.6 Expand cos x by Maclaurin's series expansion

(Gujranwala Board 2016)(Lahore Board 2016 G-I)

(Rawalpindi Board 2019)

Sol: Let $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(iv)}(x) = \cos x$

and so on …

Putting $x = 0$, we have

$f(0) = \cos 0 = 1$

$f'(0) = -\sin 0 = 0$

$f''(0) = -\cos 0 = -1$

$f'''(0) = \sin 0 = 0$

$f^{(iv)}(0) = -\cos 0 = 1$

By using Maclaurin series expansion, we have

$$f(x) = f(0) + \frac{f'(0)}{\lfloor 1}x + \frac{f''(0)}{\lfloor 2}x^2 + \frac{f'''(0)}{\lfloor 3}x^3 + \frac{f^{(iv)}(0)}{\lfloor 4}x^4 + \cdots$$

$$\cos x = 1 + \frac{0}{\lfloor 1}x + \frac{-1}{\lfloor 2}x^2 + \frac{0}{\lfloor 3}x^3 + \frac{1}{\lfloor 4}x^4 + \cdots$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$$

Q.7 Expand $f(x) = \frac{1}{1+x}$ in the Maclarin series.

(Lahore Board 2014 G-II)(Gujranwala Board 201)

Sol: Given that: $f(x) = \frac{1}{1+x} = (1+x)^{-1}$

$$f'(x) = -1(1+x)^{-2} = \frac{-1}{(1+x)^2} = \frac{-1!}{(1+x)^2}$$

$$f''(x) = (-1)(-2)(1+x)^{-3} = 2(1+x)^{-3}$$

$$= \frac{2}{(1+x)^3} = \frac{2!}{(1+x)^3}$$

$$f'''(x) = (-1)(-2)(-3)(1+x)^{-4}$$

$$= -6(1+x)^{-4} = \frac{6}{(1+x)^4} = \frac{-3!}{(1+x)^4}$$

$$f^{iv}(x) = (-1)(-2)(-3)(-4)(1+x)^{-5}$$

$$= 24(1+x)^{-5} = \frac{24}{(1+x)^5} = \frac{4!}{(1+x)^5}$$

M

Put $x = 0$, so

$f(0) = \frac{1}{1} = 1 + 0 = 1$

$f'(0) = -\frac{1!}{(1+0)^2} = -\frac{1!}{(1)^2} = -1!$

$f''(0) = \frac{2!}{(1+0)^3} = \frac{2!}{(1)^3} = 2!$

$f'''(0) = -\frac{3!}{(1+0)^4} = -\frac{3!}{(1)^4} = -3!$

$f^{iv}(0) = \frac{4!}{(1+0)^5} = \frac{4!}{(1)^5} = 4!$

M

Applying Maclaurin's series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \cdots$$

$$\frac{1}{1+x} = 1 + \frac{-1!}{1!}x + \frac{2!}{2!}x^2 + \frac{-3!}{3!}x^3 + \frac{4!}{4!}x^4 + \cdots\cdots$$

$$= 1 - x + x^2 - x^3 + x^4 + \cdots\cdots$$

Q.8 State the Taylor's series of f at x = a.

(D.G Khan Board 2017)

Sol. Taylor's Series: If f is defined in the interval containing "a" and its derivatives of all orders exist at $x = a$, then we can expand as

$$f(x) = f(a) + f'(a)(x-1) + \frac{f''(a)}{\lfloor 2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{iv}(a)}{4!}(x-a)^4 + \ldots + \frac{f^n(a)}{\lfloor n}(x-a)^n + \ldots$$

OR

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{\lfloor 2}h^2 + \frac{f'''(x)}{\lfloor 3}h^3 + \ldots + \frac{f^{(n)}(x)}{\lfloor n}h^n + \ldots$$

MULTIPLE CHOICE QUESTIONS

❑ *Each question has four possible answers. Select the correct answer and encircle it.*

Q.1 The Taylor series is valid only if it is ———
(a) Convergent (b) Divergent
(c) Increasing (d) Decreasing

Q.2 $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots =$ *(Multan Board 2013 G-II)*
(a) e^x (b) sin x
(c) $ln\ x$ (d) cos x

Q.3 $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots =$
(a) e^x (b) sin x
(c) $(1+x)^n$ (d) cos x

Q.4 The Maclaurin series expansion is valid only if it is:
(a) Convergent (b) Divergent
(c) Increasing (d) Decreasing

Q.5 $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n + \ldots\ldots$ is called: *(Gujranwala Board 2012) (Lahore Board 2017 G-II)*
(a) Maclaurine series (b) Taylor series
(c) Binomial series (d) Power series

Q.6 $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$ ——— is Maclaurines series expansion of: *(Lahore Board 2012 G-II)*
(a) cos x (b) sin x
(c) $ln\ (1-x)$ (d) $ln\ (1+x)$

Q.7 Maclaurine expansion of $ln\ (1+x)$ is: *(Bahawalpur Board 2014)*
(a) $x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots$ (b) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots$
(c) $-x - \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots$ (d) $x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots$

Q.8 sin x = *(D.G.K Board 2012 G-II)*
(a) $x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \ldots$ (b) $1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \frac{x^5}{\lfloor 6} + \ldots$
(c) $-x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots$ (d) $1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots$

Q.9 If $f(x+h) = \cos(x+h)$, then $f'(x)$ equals: *(Lahore Board 2015 G-II)*
(a) cos x (b) −cos x
(c) sin x (d) −sin x

Q.10 $1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \ldots$ is Maclaurine's series. *(Rawalpindi Board 2018)*
(a) e^x (b) sin x
(c) cos x (d) ln (1 + x)

Q.11 $1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots$ is *(Sahiwal Board 2018)*
(a) sin x (b) cos x
(c) e^x (d) e

EXERCISE 2.9

SHORT ANSWERS TO THE QUESTIONS

Q.1 Define increasing and decreasing function.
(Sargodha Board 2016)(D.G Khan Board 2017 G-I) (Multan, Lahore Board 2017 G-I)(Sargodha Board 2017)

Ans. Let f be a differentiable function on the open interval (a, b). Then
(i) f is an increasing function on (a, b) if $f'(x) > 0$ for each $x \in (a, b)$
(ii) f is an decreasing function on (a, b) if $f'(x) < 0$ for each $x \in (a, b)$

Q.2 Find the interval for which function is increasing and decreasing $f(x) = 4 - x^2, x \in (-2, 2)$
(Multan Board 2013 G-I)(Rawalpindi Board 2016) (Faisalabad Board 2017, 2019)(Multan Board 2013)

Sol: $f(x) = 4 - x^2$ Differentiating w.r.t. 'x', we have $f'(x) = -2x$

Now we will check the signs of $f'(x)$ at $x \in (-2, 2)$.

When $-2 < x < 0$, then the value of $f'(x) = -2x$ is positive. Thus at $-2 < x < 0$, the given function 'f' is an increasing function.

When $0 < x < 2$, then the value of $f'(x) = -2x$ is negative. Thus at $0 < x < 2$, the given function 'f' is a decreasing function.

Hence 'f' is an increasing function at $-2 < x < 0$ or $(-2, 0)$. and 'f' is a decreasing function at $0 < x < 2$ or $(0, 2)$.

Q.3 Find the interval for which function is increasing and decreasing $f(x) = \sin x, x \in (-\pi, \pi)$
(D.G Khan Board 2014 G-I)(Sahiwal Board 2014 G-II) (Lahore Board 2019 G-II)

Sol: $f(x) = \sin x$
Differentiating w.r.t. 'x', we have
$f'(x) = \cos x$

Now we will check the signs of $f'(x)$ at $x \in (-\pi, \pi)$.

When $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then the value of $f'(x) = \cos x$ is positive. Thus at $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the given function f is an increasing function.

When $-\pi < x < -\frac{\pi}{2}$ and $\frac{\pi}{2} < x < \pi$, then the value of $f'(x) = \cos x$ is negative. Thus at $-\pi < x < -\frac{\pi}{2}$ and $\frac{\pi}{2} < x < \pi$, the given function 'f' is a decreasing function.

Hence 'f' is an increasing function at $-\frac{\pi}{2} < x < \frac{\pi}{2}$ or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. and 'f' is a decreasing function at $-\pi < x < -\frac{\pi}{2}$ and $\frac{\pi}{2} < x < \pi$ or $\left(-\pi, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$.

Q.4 Find the interval for which function is increasing and decreasing

$f(x) = \cos x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(Rawalpindi Board 2019)

Sol: $f(x) = \cos x$

Differentiating w.r.t. 'x', we have

$f'(x) = -\sin x$

Now we will check the signs of $f'(x)$ at $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

When $-\frac{\pi}{2} < x < 0$, then the value of $f'(x) = -\sin x$ is positive. Thus at $-\frac{\pi}{2} < x < 0$, the given function 'f' is an increasing function.

When $0 < x < \frac{\pi}{2}$, then the value of $f'(x) = -\sin x$ is negative. Thus at $0 < x < \frac{\pi}{2}$, the given function 'f' is a decreasing function.

Hence 'f' is an increasing function at

$-\frac{\pi}{2} < x < 0$ or $\left(-\frac{\pi}{2}, 0\right)$

and 'f' is a decreasing function at $0 < x < \frac{\pi}{2}$ or $\left(0, \frac{\pi}{2}\right)$

Q.5 Define a stationary point.

(Faisalabad Board 2017)(Gujranwala Board 2019 G-I, II)

Sol: Any point where f is neither increasing nor decreasing is called a stationary point, provided that $f'(x)$ at that point.

Q.6 Define the critical point.

(Gujranwala Board 2013)(Lahore Board 2017 G-II) (Rawalpindi Board 2016)

Ans. Critical Point: The point where either $f'(x) = 0$ or $f(x)$ is not defined are called critical points.

Q.7 Explain relative extreme of a function.

Ans. Let $(c - \delta x, c + \delta x) \subseteq Df$ (domain of a function) where δx is small positive number.

If $f(c) \geq f(x)$ for all $x \in (c - \delta x, c + \delta x)$ then the function f is said to have relative maxima at x =c

Similarly if $f(c) \leq f(x)$ for all $x \in (c - \delta x, c + \delta x)$ then the function f has relative minima at x = c. Both relative maxima and minima are called in general relative extrema.

Q.8 Define point of inflexion. *(Bahawalpur Board 2016) (Lahore Board 2017 G-II)(Faisalabad Board 2019 G-II)*

Sol: If $f'(x) > 0$ before the point x = a, $f'(x) = 0$ at x = a and $f'(x) > 0$ after x = a or if $f'(x) < 0$ before the point x = a, $f'(x) = 0$ at x = a and $f'(x) < 0$ after x = a, then such a point of the function is called the point of inflexion.

Q.9 Explain firs derivative rule for extremum.

Sol: Let f be differentiable in neighbourhood of c where $f'(c) = 0$

1. If $f'(x)$ changes sign from positive to negative as x increases through c then f(c) is the relative maxima of f.
2. If $f'(x)$ changes sign from negative to positive as x increases through c, then f(c) is the relative minima of f.

Q.10 Explain second derivative rule for extremum.

Sol: Let f differential function in a neighbourhood of c where $f'(c) = 0$. Then

1. f has relative maxima at c if $f''(c) < 0$
2. f has relative minima if $f''(c) < 0$.

Q.11 Find the extreme values of the function

$f(x) = 5x^2 - 6x + 2$

(Gujranwala Board 2019 G-II)

Sol: $f(x) = 5x^2 - 6x + 2$ .....(i)

Differentiating w.r.t. 'x', we have

$f'(x) = 10x - 6$ .....(ii)

For critical points, put $f'(x) = 0$ we have

$10x - 6 = 0 \Rightarrow 10x = 6 \Rightarrow x = \frac{3}{5}$

Differentiating (ii) w.r.t. 'x', we have

$f''(x) = 10$

Now we will check the sign of

$f''(x)$ at $x = \frac{3}{5}$, we have

$f''\left(\frac{3}{5}\right) = 10 > 0$, therefore f has relative minima at

$x = \frac{3}{5}$

and $f\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right)^2 - 6\left(\frac{3}{5}\right) + 2 = \frac{9}{5} - \frac{18}{5} + 2$

$= \frac{9 - 18 + 10}{5} = \frac{1}{5}$

Rel Min $\left(f\left(\frac{3}{5}\right)\right) = \frac{1}{5}$

Q.12 Find the extreme values of the function

$f(x) = 5 + 3x - x^3$ *(Lahore Board 2011 G-I)*

Sol: $f(x) = 5 + 3x - x^3$ .....(i)

Differentiating w.r.t. 'x', we have

$f'(x) = 3 - 3x^2$ .....(ii)

For critical points, putting $f'(x) = 0$, we have

$3 - 3x^2 = 0 \Rightarrow 3(x^2 - 1) = 0$

$\Rightarrow \quad 3(x + 1)(x - 1) = 0$

$\Rightarrow \quad x + 1 = 0$ and $x - 1 = 0$

$\Rightarrow \quad x = -1$ and $x = 1$

Differentiating (ii) w.r.t. 'x', we have

$f''(x) = -6x$

Now we will check the sign of $f''(x)$ at $x = -1$, we have

$f''(-1) = -6(-1) = 6 > 0$, therefore 'f' has relative minima at $x = -1$

and $f(-1) = 5 + 3(-1) - (-1)^3 = 5 - 3 + 1 = 3$

Now we will check the sign of $f''(x)$ at $x = 1$, we have

$f''(1) = -6(1) = -6 < 0$, therefore 'f' has relative maxima at $x = 1$

and $f(1) = 5 + 3(1) - (1)^3 = 5 + 3 - 1 = 7$

Q.13 Examine the function defined as $f(x) = 1 + x^3$
(Lahore Board 2010 G-I)(Gujranwala Board 2010)

Sol. Given that $f(x) = 1 + x^3$

Differentiating w.r.t. 'x', we get $f'(x) = 3x^2$

$f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$

$f''(x) = 6x$ and $f''(0) = 6(0) = 0$

The second derivative does not help in determining the extreme values.

$f'(0 - \varepsilon) = 3(0 - \varepsilon)^2 = 3\varepsilon^2 > 0$

$f'(0 + \varepsilon) = 3(0 + \varepsilon)^2 = 3\varepsilon^2 > 0$

As the first derivative does not change sign at $x = 0$, therefore $(0, 0)$ is a point of inflexion.

Q.14 Show that $y = x^x$ has minimum value at

$x = \frac{1}{e}$ ***(Gujranwala Board 2012)***

(Lahore Board 2018 G-I)(Sahiwal Board 2019)

Sol: $y = x^x$

Taking λn on both sides, we have

$\lambda n\, y = \lambda n\, x^x$

$\lambda n\, y = x\, \lambda n\, x$ .....(i)

Differentiating both sides w.r.t. 'x', we have

$$\frac{1}{y}\frac{dy}{dx} = x \times \left(\frac{1}{x}\right) + \lambda n\, x \times 1$$

$$\frac{dy}{dx} = y(1 + \lambda n\, x)$$

$$\frac{dx^x}{dx} = \frac{dy}{dx} = x^x(1 + \lambda n\, x) \quad .....(ii)$$

For critical points, putting $\frac{dy}{dx} = 0$, we have

$x^x(1 + \lambda n\, x) = 0 \Rightarrow 1 + \lambda n\, x = 0 \quad (\Theta\, x^x \neq 0)$

$\Rightarrow \quad \lambda n\, x = -1 \quad \Rightarrow x = e^{-1} = \frac{1}{e}$

Differentiating (ii) w.r.t. 'x', we have

$$\frac{d^2y}{dx^2} = x^x \times \left(0 + \frac{1}{x}\right) + (1 + \lambda n\, x) \times \frac{d}{dx}(x^x)$$

$$\frac{d^2y}{dx^2} = \frac{x^x}{x} + (1 + \lambda n\, x) \times x^x(1 + \lambda n\, x) \quad (\Theta\, x^x = y)$$

$$\frac{d^2y}{dx^2} = \frac{x^x}{x} + x^x(1 + \lambda n\, x)^2$$

Now we will check the sign of $\frac{d^2y}{dx^2}$ at $x = \frac{1}{e}$, we have

$$\left.\frac{d^2y}{dx^2}\right|_{x = \frac{1}{e}} = \frac{\left(\frac{1}{e}\right)^{\frac{1}{e}}}{\left(\frac{1}{e}\right)} + \left(\frac{1}{e}\right)^{\frac{1}{e}}\left[1 + \ln\left(\frac{1}{e}\right)\right]^2$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x = \frac{1}{e}} = e\left(\frac{1}{e}\right)^{\frac{1}{e}} + \left(\frac{1}{e}\right)^{\frac{1}{e}}(1 + \lambda n\, 1 - \lambda n\, e)^2$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x = \frac{1}{e}} = e\left(\frac{1}{e}\right)^{\frac{1}{e}} + \left(\frac{1}{e}\right)^{\frac{1}{e}}(1 + 0 - 1)^2$$

$(\Theta \lambda n\, 1 = 0, \lambda n\, e = 1)$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x = \frac{1}{e}} = e\left(\frac{1}{e}\right)^{\frac{1}{e}} + 0$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x = \frac{1}{e}} = e\left(\frac{1}{e}\right)^{\frac{1}{e}} > 0 \quad (\because e^x > 0 \text{ for all values of } x)$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x = \frac{1}{e}} > 0$$

Therefore f has relative minima at $x = \frac{1}{e}$.

Q.15 Show that $y = \frac{\ln x}{x}$ has maximum value at

$x = e$ ***(Lahore Board 2012 G-II)***

Sol: $y = \frac{\ln x}{x}$ .....(i)

Differentiating w.r.t. 'x', we have

$$\frac{dy}{dx} = \frac{x \times \left(\frac{1}{x}\right) - \ln x \times 1}{x^2}\frac{dy}{dx} = \frac{1 - \ln x}{x^2} \quad .....(ii)$$

For critical points, putting $\frac{dy}{dx} = 0$, we have

$$\frac{1 - \ln x}{x^2} = 0 \Rightarrow 1 - \lambda n\, x = 0 \text{ and } x \neq 0$$

$\Rightarrow \quad \lambda n\, x = 1 \Rightarrow x = e^1 = e$

Differentiating (ii) w.r.t. 'x', we have

$$\frac{d^2y}{dx^2} = \frac{x^2 \times \left(0 - \frac{1}{x}\right) - (1 - \ln x) \times 2x}{x^4}$$

$$= \frac{-x - 2x + 2x \ln x}{x^4} = \frac{-3x + 2x \ln x}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{-3+2\ln x}{x^3}$$

Now we will check the sign of $\frac{d^2y}{dx^2}$ at $x = e$, we have

$$\left.\frac{d^2y}{dx^2}\right|_{x=e} = \frac{-3+2\ln e}{e^3}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=e} = \frac{-3+2(1)}{e^3} \qquad (\Theta \ln e = 1)$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=e} = -\frac{1}{e^3}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=e} < 0$$

Therefore 'f' has relative maxima at $x = e$.

Q.16 Determine f(x) = sin x is increasing or decreasing in the interval $\left(0, \frac{\pi}{2}\right)$.

(Faisalabad Board 2019)

Sol: $f(x) = \sin x$

$f'(x) = \cos x$

As $f'(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore$ f(x) is increasing function in $\left(0, \frac{\pi}{2}\right)$.

LONG QUESTIONS

Q.1 Show that $y = \frac{\ln x}{x}$ has maximum value at $x = e$.

(Lahore Board 2014)(Faisalabad Board 2016)

Sol: See Short Question 20

Q.2 Show that $y = x^x$ has a minimum value at $x = \frac{1}{e}$.

(D.G.K Board 2014 G-II)(Lahore Board 2018 G-I)

Sol: See Short Question 19

Q.3 Find extreme values for, $f(x) = 2x^3 - 2x^2 - 36x + 3$.

(Sargodha Board 2019)

Sol: $f(x) = 2x^3 - 2x^2 - 36x + 3$

Differentiating w.r.t "x", we have

$f'(x) = 6x^2 - 4x - 36$ .... (1)

Put $f'(x) = 0$

$\Rightarrow 6x^2 - 4x - 36 = 0 \Rightarrow 3x^2 - 2x - 18 = 0$ (Dividing by 2)

$$\Rightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 3 \times -18}}{6}$$

$$= \frac{2 \pm \sqrt{4+216}}{6} = \frac{2 \pm \sqrt{220}}{6} = \frac{2 \pm 2\sqrt{55}}{6} = \frac{1 \pm \sqrt{55}}{3}$$

Differentiating (1) w.r.t "x", we have

$f''(x) = 12x - 4$

at $x = \frac{1+\sqrt{55}}{3}$

$$f''\left(\frac{1+\sqrt{55}}{3}\right) = 12\left(\frac{1+\sqrt{55}}{3}\right) - 4$$

$$= 4\left(1+\sqrt{55}\right) - 4 = 4\left(1+\sqrt{55}-1\right)$$

$$= 4\sqrt{55} > 0$$

So f has a relative minima at $x = \frac{1+\sqrt{55}}{3}$ and its minimum value is

$$f\left(\frac{1+\sqrt{55}}{3}\right) = 2(2.81)^3 - 2(2.81)^2 - 36(2.81) + 3 \qquad \left(\Theta \frac{1+\sqrt{55}}{3} = 2.81\right)$$

$$= 44.303 - 15.79 - 101.16 + 3 = -69.65$$

at $x = \frac{1-\sqrt{55}}{3}$

$$f''\left(\frac{1-\sqrt{55}}{3}\right) = 12\left(\frac{1-\sqrt{55}}{3}\right) - 4$$

$$= 4\left(1-\sqrt{55}\right) - 4 = 4\left(1-\sqrt{55}-1\right)$$

$$= -4\sqrt{55} < 0$$

So f has a relative maxima at $x = \frac{1-\sqrt{55}}{3}$ and its maximum value is

$$f\left(\frac{1-\sqrt{55}}{3}\right) = 2(-2.14)^3 - 2(-2.14)^2 - 36(-2.14) + 3 \qquad \left(\Theta \frac{1-\sqrt{55}}{3} = -2.14\right)$$

$$= -19.58 - 9.16 + 77.04 + 3 = 51.3$$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 The function $f(x) = 3x$ has minimum value at: *(D.G.K Board 2014 G-I)*

(a) $x = 3$ (b) $x = 2$

(c) $x = 1$ (d) $x = 0$

Q.2 f(x) increases if: *(Multan Board 2013 G-I)*

(a) $f'(x) < 0$ (b) $f'(x) > 0$

(c) $f'(x) = 0$ (d) $f'(x) \geq 0$

Q.3 If $f'(C) = 0$, then if f has relative maximum at x = C: *(Gujranwala Board 2019) (Multan Board 2014 G-II, Bahawalpur Board 2014)*

(a) $f''(C) > 0$ (b) $f''(C) < 0$

(c) $f''(C) = 0$ (d) $f''(C) \geq 0$

Q.4 For a stationery point for a function f, we have f ′(x) = *(Faisalabad Board 2013)*

(a) 0 (b) +ve

(c) –ve (d) ∞

Q.5 The function $f(x) = -3x^2$ has maximum value at: *(D.G.K Board 2010)(Bahawalpur Board 2016)*

(a) x = 3 (b) x = 2

(c) x = 1 (d) x = 0

Q.5 If f(x) have second derivative at c such that f′(c) = 0 and f″(c) > 0, then c is a point of:

(a) Maxima (b) Minima

(c) Point of inflection (d) None of these

Q.7 Let f be defined on an interval (a, b) and let $x_1, x_2 \in (a, b)$. Then f is an increasing on the interval (a, b) if ——— whenever $x_2 > x_1$:

(a) $f(x_2) = f(x_1)$ (b) $f(x_2) > f(x_1)$

(c) $f(x_2) \neq f(x_1)$ (d) $f(x_2) < f(x_1)$

Q.8 Let f be a differentiable function on the interval (a, b). Then f is a decreasing on (a, b) if ——— for each $x \in (a, b)$:

(a) $f'(x) \neq 0$ (b) $f'(x) > 0$

(c) $f'(x) = 0$ (d) $f'(x) < 0$

Q.9 If $f(c) \leq f(x)$ for all $x \in (c - \delta x, c + \delta x)$, then the function f is said to have a/an ——— at x = c:

(a) Decreasing (b) Increasing

(c) Relative maxima (d) Relative minima

Q.10 Let f be differentiable in neighbourhood of c, where f′(c) = 0. If f′(x) changes sign from negative to positive as x increase through c, then f(c) is the ——— at f:

(a) Increasing (b) Decreasing

(c) Relative maxima (d) Relative minima

Q.11 Let f be differentiable function in a neighbourhood of c where f′(c) = 0. Then f has relative minima at c if f″(c) ———

(a) = 0 (b) > 0

(c) ≠ 0 (d) < 0

Q.12 Let f be defined on an interval (a, b) and let $x_1, x_2 \in (a, b)$. Then f is a/an ——— on the interval (a, b) if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$:

(a) Increasing (b) Decreasing

(c) Maxima (d) Minima

Q.13 $f'(x_1) > 0$ implies f is a/an ——— function at the point x_1:

(a) Increasing (b) Decreasing

(c) Maxima (d) Minima

Q.14 Let f be defined on an interval (a, b) and let $x_1, x_2 \in (a, b)$. Then f is a/an ——— on the interval (a, b) if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$:

(a) Increasing (b) Decreasing

(c) Maxima (d) Minima

Q.15 $f'(x_1)$ ——— implies f is a decreasing function at the point x_1:

(a) = 0 (b) < 0

(c) ≠ 0 (d) > 0

Q.16 If f(c) ——— f(x) for all $x \in (c - \delta x, c + \delta x)$, then the function f is said to have a relative minima at x = c:

(a) = (b) ≤

(c) ≠ (d) ≥

Q.17 Let f be a differentiable function on the interval (a, b). Then f is an increasing on (a, b) if ——— for each $x \in (a, b)$:

(a) $f'(x) > 0$ (b) $f'(x) < 0$

(c) $f'(x) \neq 0$ (d) $f'(x) \leq 0$

Q.18 Both relative maxima and relative minimum are called ——— in general: *(Bahawalpur Board 2018)*

(a) Relative increasing

(b) Relative decreasing

(c) Relative extreme

(d) None of these

Q.19 The maximum value of y = a sin x + b cos x is:

(a) $-\sqrt{a^2 + b^2}$ (b) $\sqrt{a^2 + b^2}$

(c) $\frac{a}{\sqrt{a^2 + b^2}}$ (d) $\frac{b}{\sqrt{a^2 + b^2}}$

Q.20 The minimum value of y = a sin x + b cos x is:

(a) $-\sqrt{a^2 + b^2}$ (b) $\sqrt{a^2 + b^2}$

(c) $\frac{a}{\sqrt{a^2 + b^2}}$ (d) $\frac{b}{\sqrt{a^2 + b^2}}$

Q.21 Let f be differential function in a neighborhood of c where f′(c) = 0. Then f has ——— at c if f″(c) < 0:

(a) Increasing (b) Decreasing

(c) Relative maxima (d) Relative minima

Q.22 Let f be differentiable in neighborhood of c, where f′(c) = 0. If f′(x) changes sign from positive to negative as x increases through c, then f(c) is the ——— at f:

(a) Increasing (b) Decreasing

(c) Relative maxima (d) Relative minima

Q.23 If $f(c) \geq f(x)$ for all $x \in (c - \delta x, c + \delta x)$, then the function f is said to have a/an ———— at x = c:
(a) Increasing (b) Decreasing
(c) Relative maxima (d) Relative minima

Q.24 Let f be differential function in a neighborhodd of c where $f'(c) = 0$. Then f has ———— at c if $f''(c) > 0$:
(a) Decreasing (b) Increasing
(c) Relative maxima (d) Relative minima

Q.25 The small change in the value of x, positive or negative is called the ———— of x:
(a) Increment (b) Differential
(c) Derivative (d) None of these

Q.26 Let f be differential function in a neighborhood of c where $f'(c) = 0$. Then f has relative maxima at c if $f''(c)$: *(Gujranwala Board 2018)*
(a) Zero (b) Negative
(c) Undefined (d) Positive

Q.27 ———— implies f is neither increasing nor decreasing function at the point x_1:
(a) $f'(x_1) = 0$ (b) $f'(x_1) > 0$
(c) $f'(x_1) < 0$ (d) None of these

Q.28 Derivative of strictly increasing function is always:
(a) Zero (b) Positive
(c) Negative (d) Both (a) and (b)

Q.29 Derivative of strictly decreasing function is always:
(a) Zero (b) Negative
(c) Positive (d) Both (a) and (b)

Q.30 The function $f(x) = 2 + 3x^2$ has minimum value at: *(Lahore Board 2013 G-I)*
(a) x = 3 (b) x = 2
(c) x = 1 (d) x = 0

Q.31 The slope of tangent line to $y = f(x)$ at (x_1, y_1) is: *(Lahore Board 2013 G-I)*
(a) m (b) $\frac{y_2 - y_1}{x_2 - x_1}$
(c) $f'(x_1)$ (d) $-\frac{dx}{dy}$

Q.32 If $f'(c) = 0$ and $f''(c) < 0$, then f(x) will given at x = c: *(Faisalabad Board 2016)*
(a) Maximum value
(b) Minimum value
(c) Neither maximum nor minimum value
(d) Stationary value

Q.33 Minimum value of the function $f(x) = x^2 + 2x - 3$ is at x = *(Gujranwala Board 2016, 2019)*
(a) –3 (b) 1
(c) 0 (d) –1

Q.34 If $f'(c) = 0$, then f(x) has relative maxima at x = c if: *(Lahore Board 2016 G-I)* *(Sargodha Board 2017)*
(a) $f''(c) < 0$ (b) $f''(c) = 0$
(c) $f''(c) > 0$ (d) $f''(c) \geq 1$

Q.35 If $f(x) = 2^x$, then $f'(x)$ equals: *(Lahore Board 2016 G-I)*
(a) 2^{x-1} (b) $2^x \ln 2$
(c) $\frac{2^x}{\ln 2}$ (d) $\frac{\ln 2}{2^x}$

Q.36 f(x) = sin x is decreasing function in the interval. *(D.G Khan Board 2017 G-I)*
(a) $\left(-\pi, -\frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, 0\right)$
(c) $\left(0, \frac{\pi}{2}\right)$ (d) $\left(-\frac{3\pi}{2}, -2\pi\right)$

Q.37 The function $f(x) = ax^2 + bx + C$ has maximum value if: *(D.G.K Board 2010)*
(a) a > 0 (b) a > 1
(c) a < 0 (d) a > 2

Q.38 The minimum value of x^x is:
(a) $(e^{-1})^{1/e}$ (b) $(e)^{1/e}$
(c) $\left(\frac{1}{e}\right)^e$ (d) None of these

Q.39 The slope of the tangent line to the graph of f defined by the equation:
y = f(x) at (x, f(x)) is
(a) f(x) (b) f'(x)
(c) $\frac{1}{f(x)}$ (d) $\frac{1}{f'(x)}$

Q.40 $\forall x \in (a, b)$, a function f(x) is said to be increasing in (a, b), if: *(Gujranwala Board 2014)*
(a) f(x) > 0 (b) f'(x) < 0
(c) f'(x) > 0 (d) f'(x) = 0

Q.41 The maximum value of $\frac{\log x}{x}$ in $0 < x < \infty$ is:
(a) e (b) $\frac{1}{e}$
(c) –e (d) $-\frac{1}{e}$

Q.42 If f(x) has second derivative at c such that $f''(c) < 0$, then c is a point of.
(Lahore Board 2018 G-I)
(a) Maxima (b) Minima
(c) Zero point (d) Point of inflection

Q.43 For relative maxima at x = c.
(Multan Board 2018 G-I)
(a) $f(c) < f(x)$ (b) $f(c) > f(x)$
(c) $f(c) \geq f(x)$ (d) $f(c) \leq f(x)$

Q.44 If $f'(a - \varepsilon) < 0$ and $f'(a + \varepsilon) < 0$, then at $x = a$ f(x) has. *(Sahiwal Board 2018)*
(Multan Board 2018 G-I)
(a) Relative minima (b) Relative maxima
(c) Point of inflection (d) Critical point

Q.45 If $f'(c) = 0$, then f(x) is minimum at $x = c$ if:
(Sahiwal Board 2019)
(a) $f''(c) > 0$ (b) $f''(c) < 0$
(c) $f''(c) = 0$ (d) $f''(c) < -1$

EXERCISE 2.10

SHORT ANSWERS TO THE QUESTIONS

Q.1 Divide 20 into two parts so that sum of their squares will be minimum.
(Lahore Board 2012 G-I)(Gujranwala Board 2018)

Sol: Let x and 20 – x be the two required positive integers such that
$x^2 + (20 - x)^2$ will be minimum.
Let $f(x) = x^2 + (20 - x)^2$ .....(i)

Differentiating w.r.t. 'x', we have

$f'(x) = 2x + 2(20 - x)^{2-1} \times (0 - 1) = 2x - 2(20 - x)$
$= 2x - 40 + 2x$
$\Rightarrow f'(x) = 4x - 40$ .....(ii)

For critical points, put $f'(x) = 0$ we have
$4x - 40 = 0 \Rightarrow 4x = 40 \Rightarrow x = 10$

Differentiating (ii) w.r.t. 'x', we have
$f''(x) = 4 \times 1 - 0 = 4 > 0$ i.e., positive

Thus f(x) gives the minimum value if $x = 10$, so the other positive integer is 10 because
$20 - x = 20 - 10 = 10$.

FORMULA FOR INTEGRATION

1. $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq -1$
2. $\int \frac{f'(x)\,dx}{f(x)} = \ln|f(x)| + c$
3. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$ $\qquad f(x) = x$, $f'(x) = 1$
4. $\int \frac{1}{x}\,dx = \ln|x| + c$
5. $\int \sin x\,dx = -\cos x + c$
6. $\int \cos x\,dx = \sin x + c$
7. $\int \sec^2 x\,dx = \tan x + c$
8. $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
9. $\int \sec x \tan^x dx = \sec x + c$
10. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$
11. $\int \tan x\,dx = \ln|\sec x| + c = -\ln|\cos x| + c$
12. $\int \cot x\,dx = \ln|\sin x| + c$
13. $\int \sec x\,dx = \ln|\sec x + \tan x| + c$
14. $\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + c$
15. $\int e^x\,dx = e^x + c$
16. $\int a^x\,dx = \frac{a^x}{\ln a} + c$
17. $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + c = -\cos^{-1}x + c$
18. $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c = \frac{-1}{a}\cot^{-1}\left(\frac{x}{a}\right)$
19. $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c = -\frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right)$

Chapter 3

INTEGRATION

EXERCISE 3.1

SHORT ANSWERS TO THE QUESTIONS

Q.1 **Find δy and dy if $y = x^2 - 1$ when x changes from 3 to 3.02.** *(Faisalabad Board 2013)*
(Multan Board 2013 G-II, 2014, 2017 G-I)
(Gujranwala Board 2017)(Sargodha Board 2019)
(Lahore Board 2011, 2012 G-I)

Ans. $y = x^2 - 1$, x changes form 3 to 3.02.

$\Rightarrow dx = \delta x = 3.02 - 3$

$dx = \delta x = 0.02$

Take differential

$dy = 2xdx$ $\qquad y = x^2 - 1$

$dy = 2(3)(0.02)$ $\qquad$ at $x = 3$

$dy = 0.12$ $\qquad y = (3)^2 - 1$

$y = x^2 - 1$ $\qquad y = 8$

$y + \delta y = (x + \delta x)^2 - 1$

$\delta y = (x + \delta x)^2 - 1 - y$

Now $\delta y = (3 + 0.02)^2 - 1 - 8$

$\delta y = 9.1204 - 9$

$\delta y = 0.1204$

Q.2 **Find δy, if $y = x^2 + 2x$ when x changes from 2 to 1.8** *(Multan Board 2014, 2018 G-II)*
(Lahore Board 2016 G-I)
(Lahore Board 2017 G-II)(Sahiwal Board 2018)

Sol: $y = x^2 + 2x$

Here $x = 2$ and $\delta x = 1.8 - 2 = -0.2$

when $x = 2$, $y = (2)^2 + 2(2) = 4 + 4 = 8$

To find δy, we have

$y + \delta y = (x + \delta x)^2 + (x + \delta x)$

$\delta + \delta y = (2 + (-0.2))^2 + 2\{2 + (-0.2)\}$

$= (1.8)^2 + 2(1.8) = 3.24 + 3.6 = 6.84$

$\delta y = 6.84 - 8 = -1.16$

$dy = (2x + 2)dx$

$= (2(2) + 2)(-0.02) = (6)(-0.02) = -1.20$

Q.3 **Find δy and dy and dy if $y = \sqrt{x}$ when x changes from 4 to 4.41.** *(D.G.K Board 2017 G-II)*
(Gujranwala Board 2013)(Faisalabad Board 2019 G-II)
(Bahawalpur Board 2016, 2019)
(Multan, Lahore Board, 2018 G-I)

Ans. $y = \sqrt{x}$; x change from 4 to 4.41

$\Rightarrow x = 4, \delta x = dx = 4.41 - 4$

$\delta x = dx = 0.41$

Take differential

$$dy = \frac{1}{2\sqrt{x}}dx$$

$$dy = \frac{1}{2\sqrt{4}}(0.41) \quad \text{at } x = 4,\ dx = 0.4$$

$$dy = \frac{0.41}{4} \approx 0.1025$$

$y = \sqrt{4}$ $\qquad$ at $x = 4$

$y = 2$

$y + \delta y = \sqrt{x + \delta x}$

$\delta y = \sqrt{x + \delta x} - y$

$\delta y = \sqrt{4 + 0.41} - 2$

$\delta y = 2.1 - 2$

$\delta y = 0.1$

Q.4 **Using differentials find $\frac{dy}{dx}$, when $xy + x = 4$**

(D.G Khan Board 2014 G-II)(Bahawalpur Board 2019)
(Lahore Board 2017 G-I)(Faisalabad Board 2017)

Sol: $xy + x = 4$

Finding differentials of both sides of the given equation, we get

$d(xy + x) = d(4)$

$d(xy) + d(x) = 0$ $\quad (\Theta\ d(f + g) = df + dg)$

$x\,dy + y\,dx + dx = 0$ ..... (1)

$\quad (\Theta\ d(f \cdot g) = f\,dg + g\,df)$

$xdy = -(y + 1)dx \Rightarrow \frac{dy}{dx} = -\frac{y + 1}{x}$

$$\frac{dy}{dx} = -\frac{y + 1}{x} \quad (x \neq 0)$$

Q.5 Using differential find $\frac{dy}{dx}$, when $x^2 + 2y^2 = 16$

(Gujranwala Board 2016, 2019 G-I, II))

Sol: Finding differentials of both sides of the given equation, we get

$d(x^2 + 2y^2) = d(16)$

$d(x^2) + d(2y^2) = 0 \quad (\Theta\ d(f+g) = df + dg)$

$2x\,dx + 2(2y\,dy) = 0 \Rightarrow 4ydy = 2x\,dx$

$2ydy = -xdx \Rightarrow dy = -\frac{x}{2y}dx \Rightarrow \frac{dy}{dx} = -\frac{x}{2y}$

Q.6 Use differentiates. Find the approximate value of $\sqrt[4]{17}$. ***(Lahore Board 2013 G-I) (Bahawalpur, Sargodha Board 2018) (Rawalpindi Board 2019)***

Ans. Let, $y = \sqrt[4]{x}$ with $x = 16$, $dx = \delta x = 1$

$y = x^{\frac{1}{4}}$

Take differential

$dx = \frac{1}{4}x^{\frac{1}{4}-1}dx$

$dy = \frac{1}{4}x^{\frac{1}{4}-1}dx$

$dy = \frac{1}{4}x^{\frac{-3}{4}}dx$

$dy = \frac{1}{4x^{\frac{3}{4}}}dx$

Put $x = 16$, $dx = 1$

$dy = \frac{1}{2(16)^{\frac{3}{4}}}(1)$

$dy = \frac{1}{2(2^4)^{\frac{3}{4}}} = \frac{1}{4(2^3)} = \frac{1}{32}$

At δy $dy = 0.03125$ and $x = 16$

$y = \sqrt[4]{16} = (2^4)^{\frac{1}{4}} = 2$

$\sqrt[4]{x + \delta x} = y + \delta y$

$\therefore \quad \sqrt[4]{17} = 2 + 0.03125$

$\sqrt[4]{17} = 2.03125$

Q.7 Use differentials, to find the value of $(31)^{1/5}$ ***(Lahore Board 2016 G-I)***

Sol: $(31)^{1/5} = (32 - 1)^{1/5}$

Take $y = x^{1/5}$ where $x = 32$ and $dx = -1$

$\frac{dy}{dx} = \frac{1}{5}x^{-\frac{4}{5}} = \frac{1}{5x^{\frac{4}{5}}}$

so $dy = \left(\frac{1}{5x^{\frac{4}{5}}}\right)dx = \left(\frac{1}{5(32)^{\frac{4}{5}}}\right)(-1) = \left(\frac{1}{5(25)^{\frac{4}{5}}}\right)(-1)$

$= \left(\frac{1}{5(2^4)}\right)(-1) = \frac{1}{5(16)}(-1) = -\frac{1}{80} = -0.0125$

Thus $(31)^{1/5} = (32)^{\frac{1}{5}} + (-0.0125) = (25)^{\frac{1}{5}} - 0.0125$

$= 2 - 0.0125 = 1.9875$

Q.8 Use differentials, to find the value of cos 29° ***(Lahore Board 2015 G-II)***

Sol: $\cos 29° = \cos(30 - 1)°$

Take $y = \cos x$, where $x = 30°$

$x + dx = 29°$, $dx = -1°$

$= -\frac{\pi}{180}$ radians $= -0.01745$ radians

$\frac{dy}{dx} = -\sin x$

$\Rightarrow \quad dy = (-\sin x)\,dx = -\sin\frac{\pi}{6}(-0.01745)$

$= \frac{1}{2}(0.01745) = 0.008725$

$\cos 29° = y + dy = \cos 30° + dy$

$= 0.866 + 0.008725 = 0.874725$

Q.9 Find the approximate increase in the volume of a cube if the length of its each edge changes from 5 to 5.02. ***(Lahore Board 2011 G-II) (Sargodha Board 2019)***

Sol: Let volume $= V = x^3$.

Here $x = 5$ and $dx = 5.02 - 5 = 0.02$

$dV = 3x^2\,dx = 3(5)2\,(0.02) = 3\,(25)(0.02)$

$= 75 \times (0.02) = 1.5$ (cubic units)

Q.10 Find δy and dy of function $f(x) = x2$ when $x = 2$ and $dx = 0.01$.

(Faisalabad Board 2019)(Rawalpindi Board 2019)

Ans. $y = x^2$ ... (1)

$y = \delta y = (x + \delta x)^2$

$\delta y = (x + \delta x)^2 - x^2$

$= x^2 + 2x\,\delta x + \delta x - x^2$

$= 2x\,\delta x + \delta x^2$

At $x = 2$, $dx = \delta x = 0.01$

$\delta y = 2(2)\,(0.01) + (0.01)^2$

$= 0.04 + 0.0001$

≈ 0.0401

Take differential on equation (1)

$dy = 2x\,dx$

$= 2(2)(0.01)$

$dy = 0.04$

LONG QUESTIONS

Q.1 Use differentials to approximate the values of $(31)^{1/5}$. *(Lahore Board 2014 G-II)*

Sol: See Short Question 9

Q.2 Use differential to approximate cos 29°. *(D.G.K Board 2014 G-I)*

Sol: See Short Question 10

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 If y = f(x) is a differentiable function, the differential of x is: *(Lahore Board 2013 G-I)*
(a) dx = 8y (b) dx = dy
(c) δx = dy (d) dx = δx

Q.2 If $y = x^2 - 1$, then dy = *(Gujranwala Board 2012 G-II)*
(a) x dx (b) (x – 1) dx
(c) 2x dx (d) 2(x – 1) dx

Q.3 The differential of x^2 is: *(Bahawalpur Board 2014)*
(a) 2x (b) 2x dx
(c) $2x\frac{dy}{dx}$ (d) $2x\frac{dx}{dy}$

Q.4 f(x + dx) = ? *(Sargodha Board 2017)*
(a) f ′ (x) dx (b) f(x) – f ′ (x) dx
(c) f(x) + f ′ (x) dx (d) –f ′ (x) dx

Q.5 If $y = x^3$, then dy = ----------. *(Lahore Board 2014 G-I)*
(a) x^3 dx (b) $3x^2$ dx
(c) $\frac{x^4}{4}$ dx (d) x dx

Q.6 Differential of y is denoted by: *(Rawalpindi Board 2019)*
(a) dy' (b) $\frac{dy}{dx}$ (c) dy (d) dx

Q.7 If $y = x^2$ then dy is: *(Sahiwal Board 2019)*
(a) 2x (b) 2x dx (c) x^2 dx (d) $2x^2$

EXERCISE 3.2

SHORT ANSWERS TO THE QUESTIONS

Q.1 Evaluate $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx \quad (x > 0)$ *(Multan Board 2013 G-I, II)(Lahore Board 2019 G-II)*

Sol: $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx = \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$

Q.2 Evaluate $\int x\left(\sqrt{x}+1\right)dx \ (x > 0)$ *(Gujranwala Board 2012)(Lahore Board 2016 G-I, 2019 G-II)*

Sol: $\int x(\sqrt{x}+1)dx = \int(x\sqrt{x}+x)\,dx$

$= \int(x^{\frac{3}{2}}+x)\,dx$

$= \int x^{\frac{3}{2}}dx + \int x\,dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{1+1}}{1+1} + c$

$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{2} + c = \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + c$

Q.3 Evaluate $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx \ (x > 0)$ *(Bahawalpur Board 2016)(Sargodha Board 2016) (Sahiwal Board 2018)*

Sol: $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$

$= \int\left[(\sqrt{x})^2 - 2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right) + \left(\frac{1}{\sqrt{x}}\right)^2\right]$

$= \int\left(x - 2 + \frac{1}{x}\right)dx$

$= \int x\,dx - 2\int dx + \int\frac{1}{x}dx$

$= \int x\,dx - 2\int 1\,dx + \int\frac{1}{x}dx = \frac{x^{1+1}}{1+1} - 2x + \ln x + c$

$= \frac{1}{2}x^2 - 2x + \ln x + c$

Q.4 Evaluate $\int\frac{3x+2}{\sqrt{x}}dx$. *(Rawalpindi Board 2013)*
(Gujranwala Board 2017)(D.G.K Board 2017 G-I) (Sahiwal, Multan Board 2014 G-II) (Sargodha Board 2019)

Ans. $\int\frac{3x+2}{\sqrt{x}}dx$

$= \int\frac{3x+2}{\sqrt{x}}dx + \int\frac{2}{\sqrt{x}}dx = 3\int\sqrt{x}\,dx + 2\int x^{\frac{-1}{2}}dx$

$= 3\frac{2^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2\frac{2^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + c = 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2x^{\frac{3}{2}} + 4\sqrt{x} + c$

Q.5 Evaluate $\int \frac{(1-\sqrt{x})^2}{\sqrt{x}}\,dx \quad (x>0)$

(Rawalpindi Board 2017 G-II)(A.J.K Board 2017) (Bahawalpur Board 2018)(Gujranwala Board 2019)

Sol: $\int \frac{(1-\sqrt{x})^2}{\sqrt{x}}\,dx \quad (x>0)$

$= \int \frac{(1)^2 - 2(1)(\sqrt{x}) + (\sqrt{x})^2}{\sqrt{x}}\,dx = \int \left(\frac{1-2\sqrt{x}+x}{\sqrt{x}}\right)dx$

$= \int \left(\frac{1}{\sqrt{x}} - 2 + \sqrt{x}\right)dx = \int \left(x^{-\frac{1}{2}} - 2 + x^{\frac{1}{2}}\right)dx$

$= \int x^{-\frac{1}{2}}\,dx - \int 2\,dx + \int x^{\frac{1}{2}}\,dx$

$= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2x + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - 2x\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = 2x^{\frac{1}{2}} - 2x + \frac{2}{3}x^{\frac{3}{2}} + c$

$= 2x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{3}{2}} - 2x + c$

Q.6 Evaluate $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}} \quad (x>0, a>0)$

(D.G.Khan Board 2017 G-I)

Sol: $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}} = \int \frac{1}{\sqrt{x+a}+\sqrt{x}} \times \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}\,dx$

$= \int \frac{\sqrt{x+a}-\sqrt{x}}{(\sqrt{x+a})^2 - (\sqrt{x})^2}\,dx = \int \frac{\sqrt{x+a}-\sqrt{x}}{x+a-x}\,dx$

$= \int \frac{\sqrt{x+a}-\sqrt{x}}{a}\,dx = \frac{1}{a}\int \left(\sqrt{x+a}-\sqrt{x}\right)dx$

$= \frac{1}{a}\left[\int \sqrt{x+a}\,dx - \int \sqrt{x}\,dx\right]$

$= \frac{1}{a}\left[\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] + c$

$= \frac{1}{a}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right] + c = \frac{1}{a}\left[\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right] + c$

$= \frac{2}{3a}\left[(x+a)^{\frac{3}{2}} - x^{\frac{3}{2}}\right] + c$

Q.7 Integrate $\int \frac{1-x^2}{1+x^2}\,dx$

(Sargodha Board 2013)(Multan Board 2013 G-I, G-II)

Ans. $\int \frac{1-x^2}{1+x^2}\,dx$

By actual division

$$\begin{array}{r} -1 \\ 1+x^2 \overline{)\,1-x^2} \\ \underline{\pm\, 1 \pm x^2} \\ 2 \end{array}$$

$\int \frac{1-x^2}{1+x^2}\,dx = \int \left(-1 + \frac{2}{1+x^2}\right)dx = \int -1\,dx + \int \frac{2}{1+x^2}\,dx$

$= -x + 2\int \frac{1}{1+x^2}\,dx$

$= -x + 2\tan^{-1}x + c \qquad \Theta \int \frac{1}{1+x^2}\,dx = \tan^{-1}x$

Q.8 Evaluate $\int (a-2x)^{3/2}\,dx$ *(Sahiwal Board 2019)*

Sol: Since $\frac{d}{dx}(a-2x) = -2$, so we write

$\int (a-2x)^{\frac{3}{2}}\,dx = -\frac{1}{2}\int (a-2x)^{\frac{3}{2}}(-2)\,dx$

$= -\frac{1}{2}\frac{(a-2x)^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$

$\left(\Theta \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1\right)$

$= -\frac{1}{2}\cdot\frac{(a-2x)^{\frac{5}{2}}}{\frac{5}{2}} + c = -\frac{1}{5}(a-2x)^{\frac{5}{2}} + c$

Q.9 Evaluate $\int \frac{ax+b}{ax^2+2bx+c}dx$.

(Sahiwal Board 2013, 2018) (Multan Board 2014, 2018 G-II)

Ans. $\int \frac{ax+b}{ax^2+2bx+c}\,dx$

$= \frac{1}{2}\int \frac{2ax+2b}{ax^2+2bx+c}\,dx = \frac{1}{2}\ln(ax^2+2bx+c) + c$

Q.10 Evaluate $\int(\cos 3x \sin 2x)\,dx$

(Sahiwal Board 2013)(Lahore Board 2017 G-I) (Lahore Board 2013 G-I, 2015 G-II) (Sargodha Board 2018)(Gujranwala Board 2019 G-I)

Ans. $\int(\cos 3x \sin 2x)dx$

$= \frac{1}{2}\int 2\cos 3x \sin 2x dx$

$(\because 2\cos\alpha \sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta))$

$= \frac{1}{2}\int[\sin(2x+2x) - \sin(3x-2x)]\,dx$

$= \frac{1}{2}\int \sin 5x dx - \frac{1}{2}\int \sin x dx$

$= \frac{1}{2}\frac{(-\cos 5x)}{5} - \frac{1}{2}(-\cos x) + c$

$= \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + c$

Q.11 Evaluate $\int \tan^2 x\,dx$

(Lahore Board 2016 G-I)(Bahawalpur Board 2016) (D.G.Khan Board 2017 G-II)(Sargodha Board 2017) (Lahore Board 2017 G-II)(Gujranwala Board 2019 G-II)

Sol: $\int \tan^2 x\,dx = \int(\sec^2 x - 1)\,dx$

$= \int \sec^2 x\,dx - \int 1\,dx = \tan x - x + c$

Q.12 Find $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}$ $(x > 0)$.

(Faisalabad Board 2013)(Lahore Board 2015 G-I)

Ans. $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}dx$

$\int\left(\frac{1}{\sqrt{x+1}-\sqrt{x}} \times \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\right)dx$

(By Rationalizing)

$\int\left(\frac{\sqrt{x+1}+\sqrt{x}}{(\sqrt{x+1})^2-(\sqrt{x})^2}\right)dx$

$\int\frac{\sqrt{x+1}+\sqrt{x}}{x+1-x}dx$

$\int(\sqrt{x+1}+\sqrt{x})\,dx$

$\int\sqrt{x+1}dx + \int\sqrt{x}dx$

$= \frac{(x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= \frac{2}{3}\left((x+1)^{\frac{3}{2}} + x^{\frac{3}{2}}\right) + C$

Q.13 Evaluate $\int \sin^2 x\,dx$

(D.G Khan Board 2014 G-I)(Bahawalpur Board 2014) (Lahore Board 2014 G-II)(Rawalpindi Board 2014) (Gujranwala Board 2013, 2014) (Faisalabad Board 2019 G-II)

Sol: $\int \sin^2 x\,dx = \int\left(\frac{1-\cos 2x}{2}\right)dx$

$= \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos 2x\,dx$

$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

Q.14 Evaluate $\int \frac{3-\cos 2x}{1+\cos 2x}dx$, $\cos 2x \neq -1$

(Lahore Board 2010 G-I)

Sol: $\int \frac{3-\cos 2x}{1+\cos 2x}dx = \int \frac{4-(1+\cos 2x)}{1+\cos 2x}dx$

$= \int \frac{4}{1+\cos 2x}dx - \int \frac{1+\cos 2x}{1+\cos 2x}dx$

$= \int \frac{4}{2\cos^2 x}dx - \int 1\,dx$

$= 2\int \sec^2 x\,dx - \int 1\,dx = 2\tan x - x + c$

MULTIPLE CHOICE QUESTIONS

Each question has four possible answers. Select the correct answer and encircle it.

Q.1 $\int \frac{1}{x^2}dx =$ *(D.G.K Board 2014 G-II)*

(a) $\ln x + C$ (b) $\ln x^2 + C$

(c) $-\frac{1}{x} + C$ (d) $\frac{-2}{x^3} + C$

Q.2 $\int(\cos^2 x + \sin^2 x)\,dx$ *(D.G.K Board 2014 G-II)*

(a) $\sin x + \cos x + C$ (b) $2\sin x + 2\cos x + C$

(c) $-\sin x + \cos x + C$ (d) $x + C$

Q.3 $\int(\frac{d}{dx}x^n)\,dx =$ *(D.G.K Board 2014 G-II) (Faisalabad Board 2018)*

(a) $\frac{x^{x+1}}{x+1}$ (b) $\frac{x^{n-1}}{n-1} + C$

(c) $n x^{n+1} + C$ (d) $x^x + C$

Q.4 For $n \neq -1$, $\int x^n\,dx =$ *(Multan Board 2014 G-I-II, D.G.K Board 2014 G-I) (Sahiwal Board 2017)*

(a) $\frac{x^{n-1}}{n-1} + C$ (b) $x^{n+1} + C$

(c) $\frac{x^{n+1}}{n+1} + C$ (d) $\frac{x^n}{n+1} + C$

Q.5 $\int \frac{f'(x)}{f(x)} dx$ =

(D.G.K Board 2014 G-I) (Multan Board 2014 G-I-II) (Lahore Board 2015 G-I)(Rawalpindi Board 2017) (Faisalabad Board 2017)(Lahore Board 2018 G-I)

(a) $ln\,|f(x)| + C$ (b) $ln\,|f'(x)| + C$

(c) $\frac{1}{f(x)} + C$ (d) $f'(x) + C$

Q.6 $\int \sin x\, dx$ = ——— *(Multan Board 2014 G-II)*

(a) $\cos x + C$ (b) $-\cos x + C$

(c) $\sin x + C$ (d) $-\sin x + C$

Q.7 $\int [f(n)]^n f'(x)\, dx$ = *(D.G.K Board 2012 G-II) (Gujranwala Board 2014, Lahore Board 2014 G-II)*

(a) $[f(x)]^{x+1} + C$ (b) $\frac{[f(x)]^{n+1}}{n+1} + C$

(c) $\frac{[f(x)]^{n+1}}{n+1}$ (d) $n[f(x)]^{n-1} + C$

Q.8 $\int e^x dx$ is equal to:

(a) $x\,e^x + C$ (b) $x\,e^{x-1} + C$

(c) $e^{x-1} + C$ (d) $e^x + C$

Q.9 $\int e^{ax} dx$ = ——— *(Bahawalpur Board 2014)*

(a) $e^{ax} + C$ (b) $e^a + C$

(c) $ae^{ax} + C$ (d) $\frac{1}{a}e^{ax} + C$

Q.10 $\int \cos 5x\, dx$ = ———

(Bahawalpur Board 2014)

(a) $5 \sin 5x + C$ (b) $\frac{1}{5}\sin 5x + C$

(c) $-5 \sin x + C$ (d) $-\frac{1}{5}\sin 5x + C$

Q.11 $\int \frac{1}{ax-1} dx$ = ——— *(Bahawalpur Board 2014)*

(a) $ln\,(ax-1) + C$ (b) $x\,ln\,(ax-1) + C$

(c) $\frac{1}{a}ln\,(ax-1) + C$ (d) $\frac{-1}{(ax-1)^2}$

Q.12 $\int \sec 5x \tan 5x\, dx$ is equal to:

(Sahiwal Board 2013, Rawalpindi Board 2014)

(a) $5 \sec 5x + C$ (b) $\frac{\sec x}{5} + C$

(c) $\frac{\sec 5x}{5} + C$ (d) $\frac{\tan 5x}{5} + C$

Q.13 $\int \cos 2x\, dx$ is equal to: *(Rawalpindi Board 2014)*

(a) $-2\sin 2x + C$ (b) $2 \sin 2x + C$

(c) $-\frac{\sin 2x}{2} + C$ (d) $\frac{\sin 2x}{2} + C$

Q.14 $\int (ax+b)^n dx$ is equal to: *(Rawalpindi Board 2014)*

(a) $\frac{(ax+b)^{n+1}}{(n+1)a} + C$ (b) $\frac{(ax+b)^{n-1}}{n-1} + C$

(c) $\frac{(ax+b)^{n+1}}{n+1} + C$ (d) $\frac{(ax+b)^n}{n+1} + C$

Q.15 $\int d(\cot ax)$ = *(Sahiwal Board 2013)*

(a) $\cot ax$ (b) $a\,cosec^2 ax$

(c) $-a\,cosec^2 ax$ (d) $\frac{cosec^2 ax}{a}$

Q.16 Differential of y is: *(Sargodha Board 2013)*

(a) dy (b) $\frac{dy}{dx}$

(c) dx (d) dy

Q.17 $\int odx$ =

(a) 1 (b) 0

(c) Constant (d) x

Q.18 $\int \frac{\sqrt{x}-1}{\sqrt{x}} dx$ *(D.G.K Board 2012 G-I)*

(a) $\frac{(\sqrt{x}-1)^2}{2} + C$ (b) $(\sqrt{x}-1) + C$

(c) $(\sqrt{x}-1)^2 + C$ (d) $\frac{1}{\sqrt{x}} - \sqrt{x} + C$

Q.19 $\int e^x dx$ equals: *(D.G.K Board 2011)*

(a) $x\,e^x + C$ (b) $x\,e^{x-1} + C$

(c) $e^x + C$ (d) $e^{x-1} + C$

Q.20 $\int (ax+b)^n dx$ equals to $(n \neq -1)$:

(D.G.K Board 2010)

(a) $\frac{n(ax+b)^{n-1}}{a} + C$ (b) $\frac{(ax+b)^{n+1}}{n+1} + C$

(c) $\frac{(ax+b)^{n+1}}{a(n+1)} + C$ (d) $\frac{(ax+b)^{n+1}}{b(n+1)} + C$

Q.21 The integration is the reverse process of:

(a) Tabulation (b) Substitution

(c) Differentiation (d) Classification

Q.22 $\int a^x dx$ equals to $(a > 0, a \neq 1)$:

(D.G.K Board 2010)

(a) $a^x + C$ (b) $-\frac{a^x}{lna} + C$

(c) $\frac{a^x}{lnx} + C$ (d) $\frac{a^x}{lna} + C$

Q.23 The inverse process of differentiation that is the process of finding such a function whose derivative is given is called.

(a) Differentiation (b) Derivative

(c) Anti-derivative (d) None of these

Q.24 An integral of $3x^2$ is:

(a) x^3 (b) 3

(c) 6x (d) x^2

Q.25 $\int \sin x\,dx =$ ----------.

(a) $\cos x + c$ (b) $-\cos x + c$

(c) $\sin x + c$ (d) $-\sin x + c$

Q.26 $\int \sec^2 x\,dx =$ ----------. *(Lahore Board 2018 G-II)*

(Faisalabad Board 2018)

(a) $\tan x + c$ (b) $-\tan x + c$

(c) $\sec x \tan x + c$ (d) $-\sec x \tan x + c$

Q.27 $\int \frac{1}{ax+b}\,dx =$ ---------- $+ c, (ax + b \neq 0, a \neq 0)$

(a) $\ln|ax + b|$ (b) $\ln|bx + a|$

(c) $a \ln|ax + b|$ (d) $\frac{1}{a} \ln|ax + b|$

Q.28 $\int [f(x)]^n f'(x)\,dx =$ ---------- $+ c, (n \neq -1)$

(a) $(n + 1) f(x)]^{n+1}$ (b) $f(x)$

(c) $n f(x)^{n-1}$ (d) $\frac{[f(x)]^{n+1}}{n+1}$

Q.29 $\int \cos x\,dx =$ -------- + c.

(a) $\cos x$ (b) $-\cos x$

(c) $\sin x$ (d) $-\sin x$

Q.30 $\int \{f(x) - g(x)\}\,dx =$

(a) $\int f(x)\,dx + \int g(x)\,dx$ (b) $\int f(x)\,dx - \int g(x)\,dx$

(c) $\int f(x)\,dx \times \int g(x)\,dx$ (d) $\int f(x)\,dx \pm \int g(x)\,dx$

Q.31 If $\phi'(x) = f(x)$ and $\int_a^b f(x)\,dx$ has a definite value $\phi(b) - \phi(a)$, then it is called the ---------- of f from a to b.

(a) Integration by parts (b) Definite integral

(c) Differentiation (d) None of these

Q.32 $\int \frac{1}{1 - \sin^2 x}\,dx =$

(a) $\sec^2 x + c$ (b) $\sec x + c$

(c) $\cos x + c$ (d) $\tan x + c$

Q.33 $\int \{f(x) + g(x)\}\,dx =$

(a) $\int f(x)\,dx + \int g(x)\,dx$ (b) $\int f(x)\,dx - \int g(x)\,dx$

(c) $\int f(x)\,dx \times \int g(x)\,dx$ (d) $\int f(x)\,dx \pm \int g(x)\,dx$

Q.34 $\int [f(x)]^{-1} f'(x)\,dx =$ ----------.

(Gujranwala Board 2014)

(a) $\ln|f(x)| + c$ (b) $-\ln|f(x)| + c$

(c) $\ln|f'(x)| + c$ (d) $-\ln|f'(x)| + c$

Q.35 $\int [f(x)]^n f'(x)\,dx =$ ----------

(Gujranwala Board 2014)

(a) $[f(x)]^{n+1} + c$ (b) $\frac{[f(x)]^{n+1}}{n+1}$

(c) $[f(x)]^{n+1}(n + 1) + c$

(d) $n[f(x)]^{n-1} + c$

Q.36 $\int \frac{1}{x}\,dx =$ -------- *(Gujranwala Board 2014)*

(D.G.K Board 2010)(Bahawalpur Board 2018)

(a) $\ln\left|\frac{1}{x}\right| + c$ (b) $\ln|x| + c$

(c) $\frac{x^{-1+1}}{-1+1} + c$ (d) $-1\,x^{-2} + c$

Q.37 $\int \sin x \cos x\,dx =$ *(Lahore Board 2018 G-II)*

(a) $\frac{1}{2} \cos 2x$ (b) $-\frac{1}{2} \cos 2x$

(c) $\frac{\sin^2 x}{2}$ (d) $\frac{\cos^2 x}{2}$

Q.38 Anti-derivative of cot x is equal. *(Lahore Board 2014 G-I)*

(a) $\ln \cos x + c$ (b) $\ln \sin x + c$

(c) $-\ln \cos x + c$ (d) $-\ln(\sin x) + c$

Q.39 $\int \cos^2 ax\,dx =$ *(Lahore Board 2013 G-II)*

(a) $\cos^3\left(\frac{ax}{3}\right) + c$ (b) $\frac{1}{3}\cos^3(ax) + c$

(c) $\frac{x}{2} + \frac{\sin 2ax}{4a} + c$ (d) $\frac{\cos ax}{2a} + c$

Q.40 $\int_\alpha^x 3t^2\,dt =$ *(Lahore Board 2013 G-I)*

(a) $\alpha^3 - x^3$ (b) $\alpha^3 + x^3$

(c) $x^3 - \alpha^3$ (d) $\frac{x^3 + \alpha^3}{3}$

Q.41 $\int \frac{\sin 2x}{\sin x}\,dx$

(a) $\sin 2x$ (b) $2\sin 2x$

(c) $\frac{1}{2} \sin x$ (d) $2\sin x$

Q.42 $\int \frac{dx}{\sqrt{a^2 - x^2}}$ *(D.G.K Board 2012 G-I)*

(a) $\tan^{-1} x + C$ (b) $\sin^{-1} \frac{x}{a} + C$

(c) $\cos^{-1} \frac{x}{a} + C$ (d) None

Q.43 $\int \frac{x^2+2x+1}{\sqrt{x}}\,dx =$

(a) $\frac{(x^2+2x+1)^2}{2}$

(b) $\frac{2}{5}x^{5/2}+\frac{4}{3}x^{3/2}+\sqrt{x}+c$

(c) $\frac{5}{2}x^{5/2}+\frac{3}{2}x^{3/2}+x^{1/2}+c$

(d) $\frac{2}{5}x^{5/2}+\frac{4}{3}x^{3/2}+2x^{1/2}+c$

Q.44 $\int (2x)^{\frac{3}{2}}\,dx =$ *(D.G.K Board 2017 G-I)*

(a) $\frac{1}{5}(2x)^{\frac{5}{2}}+C$ (b) $\frac{2}{5}(2x)^{\frac{5}{2}}+C$

(c) $\frac{1}{2}(2x)^{\frac{3}{2}}+C$ (d) $\frac{2}{3}(2x)^{\frac{1}{2}}+C$

Q.45 $\int \sin 2x\,dx$ *(Rawalpindi Board 2017 G-II)*

(a) $-\frac{\cos 2x}{2}$ (b) $\frac{\cos 2x}{2}$

(c) $2\cos 2x$ (d) $-2\cos 2x$

Q.46 $\int x(\sqrt{x}+1)\,dx$ *(Sahiwal Board 2017)*

(a) $\frac{2}{3}x^{\frac{3}{2}}+C$ (b) $\frac{2}{5}x^{\frac{5}{2}}+C$

(c) $\frac{2}{5}x^{\frac{5}{2}}+\frac{x^2}{2}+C$ (d) $x^{\frac{3}{2}}+x+C$

Q.47 $\int \sin 3x\,dx =$

(a) $\frac{\cos 3x}{3}$ (b) $3\cos 3x+C$

(c) $-3\cos 3x+C$ (d) $\frac{-\cos 3x}{3}+C$

Q.48 $\int 2^x\,dx =$ *(Gujranwala Board 2018)*

(a) $\frac{2^{x+1}}{x+1}$ (b) $x\,2^{x-1}$

(c) $2^x \ln 2$ (d) $\frac{2^x}{\ln 2}$

Q.49 $\int 3^x\,dx =$ *(Sargodha Board 2018)*

(a) 3^x+C (b) $3^x \ln 3+C$

(c) $\frac{3^x}{\ln 3}$ (d) $3\ln 3^x+C$

Q.50 $\int \frac{1+x}{x}\,dx$ *(Sargodha Board 2018)*

(a) $\log_e|x|+C$ (b) $1+\log_e|x|+C$

(c) $\log_e|1+x|$ (d) $x+\log_e|x|+C$

Q.51 If $\int a^x\,dx =$ *(D.G.K Board 2014 G-I) (Sahiwal Board 2017)*

(a) $a^x \ln a+C$ (b) a^x+C

(c) $\frac{a^x}{x}+C$ (d) $\frac{a^x}{\ln a}+C$

Q.52 $\int x^{-1}\,dx =$: *(Faisalabad Board 2019 G-II)*

(a) $\lambda nx+c$ (b) $\frac{x^{-2}}{2}$

(c) $-x^2$ (d) 0

Q.53 Anti derivative of cot x or $\int \cot x\,dx$.

(Sargodha Board 2019)(Gujranwala Board 2019 G-I, II)

(a) $\lambda n \cos x+c$ (b) $\lambda n \sin x+c$

(c) $-\lambda n \cos x+c$ (d) $-\lambda n \sin x+c$

Q.54 $\int \tan^2 x\,dx$ is equal to:

(Gujranwala Board 2019 G-II)

(a) $\tan x+x+c$ (b) $\tan x-x+c$

(c) $2\tan x+c$ (d) $2\tan x+x+c$

Q.55 $\int \sec^2 x dx =$: *(Lahore Board 2019)*

(a) $\cot x+c$ (b) $\tan x+c$

(c) $2\sec x+c$ (d) $\frac{1}{\cos^2 x}+c$

Q.56 $\int \frac{\sin 2x}{4\sin x}dx =$ *(Multan Board 2019)*

(a) $\sin 2x+c$ (b) $2\sin 2x+c$

(c) $\frac{1}{2}\sin x+c$ (d) $2\sin x+c$

Q.57 $\int 3\sin 3x dx =$ *(Sahiwal Board 2019)*

(a) $\cos 3x$ (b) $-\cos 3x$

(c) $a\sin 3x$ (d) $9\cos 3x$

EXERCISE 3.3

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find $\int \frac{dx}{x^2+4x+13}$

(Gujranwala Board 2013) (Sahiwal Board 2014 G-I)

Ans. $\int \frac{dx}{x^2+4x+13}$

$= \int \frac{dx}{x^2+4x+4+9} = \int \frac{dx}{(x+2)^2+(3)^2}$

$= \frac{1}{3}\tan^{-1}\frac{x+2}{3}+c$

Q.2 Evaluate $\int \frac{x^2}{4+x^2}\,dx$.

(Sahiwal Board 2013, 2018)(Sargodha Board 2013) (Lahore Board 2011 G-II , 2012 G-I) (Bahawalpur Board 2016)(Gujranwala Board 2017) (Faisalabad Board 2019 G-I)

Ans. $\int \frac{x^2}{4+x^2}\,dx.$

$= \int \frac{4+x^2-4}{4+x^2}\,dx = \int \frac{4+x^2}{4+x^2}\,dx - 4\int \frac{1}{x^2+(2)^2}\,dx$

$= \int 1dx - 4\int \frac{1}{x^2+(2)^2}dx = x - 4\frac{1}{2}\tan^{-1}\frac{x}{2} + c$

$= x - 2\tan^{-1}\frac{x}{2} + c$

Q.3 Find $\int \frac{1}{x\ln x}\,dx$.

(Faisalabad Board 2013) (Lahore Board 2016 G-I) (Gujranwala Board 2016)(Bahawalpur Board 2018) (Multan Board 2018 G-I, II)(Sargodha Board 2019)

Ans. $\int \frac{1}{x\ln x}\,dx.$

$= \int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx$

$= \ln(\ln x) + C$

Q.4 $\int \frac{e^x}{e^x+3}\,dx$ *(Gujranwala Board 2018)*

(Bahawalpur Board 2014)(Multan Board 2013) (Sargodha Board 2019)

Ans. $\int \frac{e^x}{e^x+3}$

$t = e^x + 3$

$dt = e^x\,dx = \int \frac{dt}{t} = \ln|t| + C = \ln|e^x+3| + C$

Q.5 Evaluate $\int \frac{(x+b)}{(x^2+2bx+c)^{\frac{1}{2}}}\,dx.$

(Rawalpindi Board 2013)(Lahore Board 2018 G-II)

Ans. $\int \frac{(x+b)}{(x^2+2bx+c)^{\frac{1}{2}}}dx$

$= \frac{1}{2}\int (x^2+2bx+c)^{\frac{-1}{2}}(2x+2b)\,dx$

$= \frac{1}{2}\frac{(x^2+2bx+c)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + c$

$= \frac{1}{2}\frac{\sqrt{x^2+2bx+c}}{\frac{1}{2}} + c = \sqrt{x^2+2bx+c} + c'$

Q.6 Evaluate $\int \frac{\sin\theta}{1+\cos^2\theta}\,d\theta$ *(Lahore Board 2013)*

(Gujranwala Board 2019)

Ans. $\int \frac{\sin\theta}{1+\cos^2\theta}\,d\theta$

Let, $\cos\theta = t$

$-\sin\theta\,d\theta = dt$

$\sin\theta\,d\theta = -dt$

$\int \frac{-dt}{1+t^2}$

$= \int \frac{1}{1+t^2}dt = -\tan^{-1}(t) + c = -\tan^{-1}(\cos\theta) + c$

Q.7 $\int \sec x\,dx$ *(Rawalpindi Board 2017 G-II)*

(Murltan Board 2013 G-I)

Ans. $\int \sec x\,dx$

$= \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx$

$u = \sec x + \tan x$

$du = (\sec x\tan x + \sec^2 x)\,dx$

$= \sec x(\tan x + \sec x)\,dx$

$= \int \frac{du}{u} = \ln|u| + c = \ln|\sec x + \tan x| + c$

Q.8 Evaluate $\int \operatorname{cosec} x\,dx$

(Multan Board 2014 G-I)(Sargodha Board 2018)

Sol: $\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x}dx$

$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x\cot x}{\operatorname{cosec} x - \cot x}dx$

Put $u = \operatorname{cosec} x - \cot x$

$\Rightarrow du = (-\operatorname{cosec} x\cot x + \operatorname{cosec}^2 x)\,dx$

$\int \operatorname{cosec} x\,dx = \ln(\operatorname{cosec} x - \cot x) + c$

Q.9 Evaluate $\int \sqrt{\frac{1+x}{1-x}}\,dx$

Sol: $\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}}\times\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx$

(Rationalization)

$= \int \frac{(\sqrt{1+x})^2}{\sqrt{(1-x)(1+x)}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx$

$$= \int\left(\frac{1}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}\right)dx$$

$$= \int\frac{1}{\sqrt{1-x^2}}dx+\int\frac{x}{\sqrt{1-x^2}}dx$$

$$= \sin^{-1}x-\frac{1}{2}\int(1-x^2)^{-1/2}(-2x)\,dx$$

$$= \sin^{-1}x-\frac{1}{2}\frac{(1-x^2)^{1/2}}{\frac{1}{2}}+c$$

$$= \sin^{-1}x-\sqrt{1-x^2}+c$$

Q.10 Evaluate $\int\frac{\cot\sqrt{x}}{\sqrt{x}}dx$

(Rawalpindi Board 2016)(Sargodha Board 2016) (Faisalabad Board 2017)(D.G.Khan Board 2017 G-I,II)

Sol: Putting $\sqrt{x} = t \Rightarrow \frac{1}{2}x^{-\frac{1}{2}}dx = dt$

$$\frac{dx}{\sqrt{x}} = 2dt$$

Therefore, $\int\frac{\cot\sqrt{x}}{\sqrt{x}}dx = \int\cot t\cdot 2\,dt = 2\int\frac{\cos t}{\sin t}dt$

$$= 2\ln(\sin t)+c = 2\ln(\sin\sqrt{x})+c$$

Q.11 Evaluate $\int\frac{\cos x}{\sin x\,\lambda n\sin x}dx$

(Lahore Board 2013 G-II)(Sahiwal Board 2013) (D.G. Khan Board 2017 G-II)(Balawalpur Board 2019)

Sol: Let $I = \int\frac{\cos x}{\sin x\,\lambda n\sin x}dx$

$$= \int\frac{1}{\lambda n\sin x}\left(\frac{\cos x}{\sin x}\right)dx$$

Then $I = \int\frac{1}{4}\,du$

$= \ln|u| + c$

$= \ln|\ln|\sin x|| + c$

Putting $u = \lambda n\sin x$,

then $du = \frac{1}{\sin x}\cos x\,dx$

$$= \left(\frac{\cos x}{\sin x}\right)dx$$

Q.12 Evaluate $\int\cos x\left(\frac{\lambda n\sin x}{\sin x}\right)dx$

Sol: Let $I = \int\cos x\left(\frac{\lambda n\sin x}{\sin x}\right)dx$

$$= \int\lambda n\sin x\left(\frac{\cos x}{\sin x}\right)dx$$

Putting $u = \lambda n\sin x$,

then $du = \frac{1}{\sin x}\cos x\,dx$

$$= \frac{\cos x}{\cos x}dx$$

$I = \int u\,du = \frac{u^2}{2}+c = \frac{1}{2}(\lambda n\sin x)^2+c$

Q.13 Evaluate $\int\frac{dx}{\sqrt{a^2-x^2}}$

(D.G Khan Board 2014 G-I)(Faisalabad Board 2019 G-II)

Sol: Let $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

Therefore, $\int\frac{dx}{\sqrt{a^2-x^2}} = \int\frac{a\cos\theta\,d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$

$$= \int\frac{a\cos\theta}{a\sqrt{1-\sin^2\theta}}d\theta$$

$$= \int\frac{\cos\theta}{\cos\theta}d\theta = \int d\theta = \theta+c$$

$$= \sin^{-1}\left(\frac{x}{a}\right)+c$$

Q.14 Evaluate $\int\frac{\sec^2 x}{\sqrt{\tan x}}dx$

(Lahore Board 2014 G-II)(Multan Board 2014 G-I) (Sargodha Board 2018)

Sol: $\int\frac{\sec^2 x}{\sqrt{\tan x}}dx = \int(\tan x)^{-\frac{1}{2}}(\sec^2 x)\,dx$

$$= \frac{(\tan x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c = \frac{(\tan x)^{\frac{1}{2}}}{\frac{1}{2}}+c = 2\sqrt{\tan x}+c$$

Q.15 Evaluate $\int\frac{\sqrt{2}}{\sin x+\cos x}dx$

(Lahore Board 2014 G-I) (Faisalabad Board 2019 G-II) (Gujranwala Board 2019 G-II)

Sol: $\int\frac{\sqrt{2}}{\sin x+\cos x}dx = \int\frac{1}{\frac{\sin x+\cos x}{\sqrt{2}}}dx$

$$= \int \frac{1}{\sin x \times \frac{1}{\sqrt{2}} + \cos x \times \frac{1}{\sqrt{2}}} dx$$

$$= \int \frac{1}{\sin x \sin\frac{\pi}{4} + \cos x \cos\frac{\pi}{4}} dx$$

$$= \int \frac{1}{\cos\left(x - \frac{\pi}{4}\right)} dx = \int \sec\left(x - \frac{\pi}{4}\right) dx$$

$$= \lambda n \left| \sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right) \right| + c$$

Q.16 Find $\int \frac{dx}{x(\ln 2x)^3}$ $(x > 0)$.

(Lahore Board 2013, 2015 G-I)

Ans. $\int \frac{dx}{x(\ln 2x)^3}$

Let $\ln 2x = t$

$\frac{1}{2x} \cdot 2dx = dt$

$\frac{1}{2x} \cdot 2dx = dt$

$\frac{1}{x} dx = dt$

So, $\int \frac{1}{t^3} dt = \int t^{-3} dt$

$= \frac{t^{-3+1}}{-3+1} + c = \frac{(\ln 2x)^{-2}}{-2} + c = -\frac{1}{2(\ln x)^2} + c$

Q.17 Evaluate $\int x\sqrt{x^2-1}\, dx$

(Lahore Board 2015 G-II)(Rawalpindi Board 2016) (Sargodha Board 2016,2017)(Gujranwala Board 2017) (Faisalabad Board 2017)

Sol: $\int x\sqrt{x^2-1}\, dx = \int (x^2-1)^{\frac{1}{2}} x\, dx$

$$= \frac{1}{2}\int (x^2-1)^{\frac{1}{2}} (2x)\, dx = \frac{1}{2}\left\{\frac{(x^2-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right\} + c$$

$$= \frac{1}{2}\left\{\frac{(x^2-1)^{\frac{3}{2}}}{\frac{3}{2}}\right\} + c = \frac{1}{3}\left[x^2-1\right]^{\frac{3}{2}} + c$$

LONG QUESTIONS

Q.1 Evaluate $\int \sqrt{\frac{1+x}{1-x}}\, dx$. *(Lahore Board 2013 G-II)*

Sol: See Short Question 13

Q.2 Evaluate $\int \frac{\cos x}{\sin x \ln(\sin x)} dx$.

(Lahore Board 2016 G-I)

Sol: See Short Question 17

Q.3 Show that $\int \frac{dy}{\sqrt{x^2-a^2}} = \ln\left(x + \sqrt{x^2-a^2}\right) + c$.

(Faisalabad Board 2013)(Gujranwala Board 2013) (Lahore Board 2015 G-I, 2017, 2018 G-II) (Sargodha Board 2019)

Sol: $\int \frac{1}{\sqrt{x^2-a^2}} dx$

Put $x = a \sec\theta \Rightarrow dx = a \sec\theta \tan\theta\, d\theta$

$\int \frac{1}{\sqrt{x^2-a^2}} dx = \int \frac{1}{\sqrt{a^2\sec^2\theta - a^2}} \cdot a\sec\theta \cdot \tan\theta\, d\theta$

$= \int \frac{1}{\sqrt{a^2(\sec^2\theta - 1)}} \cdot a\sec\theta\tan\theta\, d\theta$

$= \int \frac{1}{\sqrt{a^2\tan^2\theta}} \cdot a\sec\theta \cdot \tan\theta\, d\theta$

$= \int \frac{1}{a\tan\theta} \cdot a\sec\theta \cdot \tan\theta\, d\theta$

$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c'$

$= \ln\left|\frac{x}{a} + \sqrt{\sec^2\theta - 1}\right| + c' \qquad (\Theta\ x = a\sec\theta)$

$= \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + c' = \ln\left|\frac{x}{a} + \sqrt{\frac{x^2-a^2}{a^2}}\right| + c'$

$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + c' = \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c'$

$= \ln(x + \sqrt{x^2-a^2}) - \ln a + c'$

$= \ln\left(x + \sqrt{x^2-a^2}\right) + c \qquad (\Theta -\ln a + c' = c)$

Q.4 Show that

$$\int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{2} + c.$$

(Gujranwala Board 2012)(Lahore Board 2014 G-I) (Faisalabad Board 2016)

Sol: $\int \sqrt{(a^2-x^2)}\, dx$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

$\int \sqrt{a^2-x^2}\, dx = \int \sqrt{a^2 - a^2\sin^2\theta}\ (a\cos\theta)\, d\theta$

$= \int \sqrt{a^2(1-\sin^2\theta)}\ \ (a\cos\theta)\ d\theta$

$= \int \sqrt{a^2(\cos^2\theta)}\,(a\cos\theta)d\theta = \int a\cos\theta \,.\, a\cos\theta d\theta$

$= a^2 \int \cos^2\theta d\theta = a^2 \int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$

$(\Theta \cos^2\theta = \frac{1+\cos 2\theta}{2})$

$= \frac{a^2}{2}\int 1\, d\theta + \frac{a^2}{2}\int \cos 2\theta d\theta$

$= \frac{a^2}{2}\theta + \frac{a^2}{2}\frac{\sin 2\theta}{2} + c = \frac{a^2}{2}\theta + \frac{a^2}{4}[\sin 2\theta] + c$

$= \frac{a^2}{2}\theta + \frac{a^2}{4}[2\sin\theta\cos\theta] + c$

$= \frac{a^2}{2}\theta + \frac{a^2}{2}[\sin\theta\sqrt{1-\sin^2\theta}] + c$

$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{2}\left[\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right] + c$

$(\Theta\, x = a\sin\theta)$

$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{ax}{2}\sqrt{\frac{a^2-x^2}{a^2}} + c$

$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{ax}{2}\cdot\frac{\sqrt{a^2-x^2}}{a} + c$

$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 $\int \frac{\text{cosec}^2 x}{\cot x} dx$ equals: *(Sahiwal Board 2014)*

(a) $\tan x + A$ (b) $ln \cot x + C$

(c) $\cot x + C$ (d) $-ln \cot x + C$

Q.2 $\int \cot x\, dx$ equals: *(Sahiwal Board 2014, 2017) (D.G.K Board 2014 G-I)(Rawalpindi Board 2017 G-I)*

(a) $ln \sec x + C$ (b) $ln \text{cosec}\, x + C$

(c) $ln \sin x + C$ (d) $ln \cot x + C$

Q.3 $\int \sec x \tan x\, dx$ = *(Lahore Board 2016 G-I) (D.G.K Board 2014 G-II) (Sahiwal Board 2015 G-I)*

(a) $\sec^2 x + C$ (b) $\tan x + C$

(c) $\sec x^2 + C$ (d) None of these

Q.4 $\int \frac{\log_e \tan x}{\sin 2x} dx$ = *(Multan Board 2018 G-I)*

(a) $\frac{1}{2}\log_e \tan x + C$ (b) $\frac{1}{4}(\log_e(\tan x))^2 + C$

(c) $\frac{1}{2}\log_e(\sin 2x)^2 + C$ (d) $\frac{1}{4}\log_e(\sin 2x)^2 + C$

Q.5 $\int \frac{e^{\tan^{-1}x}}{1+x^2} dx$

(Lahore Board 2014 G-II, Multan Board 2014 G-II)

(a) $e^{\sec x} + C$ (b) $e^{\tan x} + C$

(c) $e^{\cot^{-1}x} + C$ (d) $e^{\tan^{-1}x} + C$

Q.6 $\int \cot x\, dx$ equals: *(Multan Board 2014 G-II) (D.G.K Board 2011)*

(a) $-\text{cosec}^2 x + c$ (b) $\text{cosec}^2 x + C$

(c) $ln \cos x + C$ (d) $ln \sin x + C$

Q.7 The anti-derivative of $\frac{1}{(1+x^2)\tan^{-1}x}$ equals:

(Multan Board 2014 G-I)(D.G.K Board 2011)

(a) $ln(\tan^{-1}x) + C$ (b) $ln(\tan x) + C$

(c) $\tan^{-1}x + C$ (d) $\tan x + C$

Q.8 Suitable substitution for $\int \frac{1}{x\sqrt{x^2-a^2}} dx$ is:

(Sahiwal Board 2013)(Lahore Board 2018 G-II)

(a) $x = a\tan\theta$ (b) $x = a\sec\theta$

(c) $x = a\sin\theta$ (d) $x = a\cos\theta$

Q.9 $\int \frac{-1}{x\sqrt{x^2-1}} dx$ = *(Lahore Board 2014 G-II) (Gujranwala Board 2012)*

(a) $\tan^{-1}x + C$ (b) $\text{cosec}^{-1}x + C$

(c) $\sec^{-1}x + C$ (d) $\sin^{-1}x + C$

Q.10 $\int e^{(\lambda x+\mu)} dx$ = *(Sahiwal Board 2013)*

(a) $\frac{e^{\lambda x+\mu}}{\lambda} + C$ (b) $\frac{1}{\mu}e^{\lambda x+\mu} + C$

(c) $\lambda e^{\lambda x+\mu} + C$ (d) $\mu e^{\lambda x+\mu} + C$

Q.11 $\int \frac{e^x}{e^x+3} dx$ = *(Sahiwal Board 2013)*

(a) $ln(e^x+3) + C$ (b) $ln\, e^x + C$

(c) $3\, ln(e^x+3) + C$ (d) $\frac{1}{3}ln(e^x+3) + C$

Q.12 $\int \cos\left(\sqrt{x}-\frac{x}{2}\right)\times\left(\frac{1}{\sqrt{x}}-1\right) dx$ =

(Sahiwal Board 2013)(Rawalpindi Board 2017)

(a) $2\cos\left(\sqrt{x}-\frac{x}{2}\right)+C$ (b) $2\sin\left(\sqrt{x}-\frac{x}{2}\right)+C$

(c) $2\cos\left(\frac{1}{x}-1\right)+C$ (d) $2\cos\frac{1}{x}+C$

Q.13 $\int \frac{1}{x^2+9} dx$ = *(Rawalpindi Board 2014)*

(a) $\frac{1}{9}\tan^{-1}\frac{x}{3} + C$ (b) $\frac{1}{3}\tan^{-1}\frac{x}{3} + C$

(c) $\frac{1}{9}\tan^{-1}\frac{x}{9} + C$ (d) $\frac{1}{3}\tan^{-1}\frac{x}{9} + C$

Q.14 $\int \left(x+\frac{1}{x}\right)^{1/2}\left(1-\frac{1}{x^2}\right)dx$ =

(Rawalpindi Board 2014)

(a) $\frac{\left(\left(x+\frac{1}{x}\right)\right)^{3/2}}{2}+C$ (b) $\frac{\left(\left(x+\frac{1}{x}\right)\right)^{3/2}}{3}+C$

(c) $\frac{2}{3}\left(x+\frac{1}{x}\right)^{3/2}+C$ (d) $\frac{3}{2}\left(x+\frac{1}{x}\right)^{3/2}+C$

Q.15 Suitable substitution for $\int \frac{1}{\sqrt{a^2-x^2}}dx$ is:

(a) $x = a\sin^2\theta$ (b) $x = a\cos\theta$

(c) $x = a\sec\theta$ (d) $x = a\tan\theta$

Q.16 $\int \frac{1}{x\sqrt{x^2-a^2}}dx$ = *(Multan Board 2013 G-II)*

(a) $\frac{1}{a}\sec^{-1}\frac{x}{a}+C$ (b) $\frac{1}{a}\text{cosec}^{-1}\frac{x}{a}+C$

(c) $\frac{1}{a}\tan^{-1}\frac{x}{a}+C$ (d) $\frac{1}{a}\cot^{-1}\frac{x}{a}+C$

Q.17 $\int (2x+3)^{1/2}dx$ = *(Multan Board 2013 G-II)*

(a) $\frac{(2x+3)^{3/2}}{3}+C$ (b) $\frac{(2x+3)^{1/2}}{2}+C$

(c) $\frac{(2x+3)^{1/2}}{3/2}+C$ (d) $\frac{(2x+3)^{1/2}}{2/3}+C$

Q.18 $\int \sec x\,dx$ equals: *(Lahore Board 2017 G-I)*
(Multan 2013 G-I, Lahore 2014 G-II, Gujranwala 2014)

(a) $\ln|\sec x + C$ (b) $\ln|\sec x + \tan x| + C$

(c) $\sec x \tan x + C$ (d) $\sec x + \tan x + C$

Q.19 $\int \frac{\sec^2 x\,dx}{\sqrt{\tan x}}$ equals: *(Multan Board 2013 G-I)*
(Sahiwal Board 2018)

(a) $\sqrt{\tan x}+C$ (b) $2\sqrt{\tan x}+C$

(c) $\sqrt{\cos x}+C$ (d) $2\sqrt{\cot x}+C$

Q.20 $\int \tan x\,dx$ = *(Sargodha Board 2013)*
(Lahore Board 2017 G-I)(Lahore Board 2012 G-I)
(Faisalabad Board 2017)(A.J.K Board 2017)

(a) $\ln|\cos x| + C$ (b) $\ln|\sec x| + C$

(c) $\ln|\text{cosec } x| + C$ (d) $\ln|\cot x| + C$

Q.21 $\int \left(\frac{1}{x}+\frac{\sin 2x}{\sin^2 x}\right)dx$ = *(Sargodha Board 2013)*

(a) $\ln|x\tan^2 x|$ (b) $\ln|x\sin^2 x| + C$

(c) $\ln|e^x\cos^2 x| + C$ (d) $\ln|x\cot^2 x|$

Q.22 $\int a^{x^2}x\,dx$ *(Sargodha Board 2013, 2018)*

(a) $a^{x^2}\frac{\lambda}{\ln a}+C$ (b) $\frac{a^{x^2}}{2\ln a}+C$

(c) $a^{x^2}+C$ (d) $2a^{x^2}+C$

Q.23 $\int \left(1-\frac{\sin 2x}{\cos^2 x}\right)dx$ is equal to:

(Rawalpindi Board 2013)

(a) $\ln|x\sin x| + C$ (b) $\ln|x\sin^2 x| + C$

(c) $\ln|e^x\cos^2 x| + C$ (d) $\ln|x\cos^2 x| + C$

Q.24 $\int \frac{dx}{\sqrt{a^2+x^2}}$ is equal to: *(Rawalpindi Board 2013)*

(a) $\tan^{-1}\frac{x}{a}+C$ (b) $\cot^{-1}\frac{x}{a}+C$

(c) $\sin^{-1}\frac{x}{a}+C$ (d) $\ln\left|x+\sqrt{a^2+x^2}\right|$

Q.25 $\int \sec^2 x\,dx$ = *(Faisalabad Board 2013)*

(a) $\frac{n}{3}\sec 3nx + C$ (b) $n\tan nx + C$

(c) $\tan nx + C$ (d) $\frac{1}{n}\tan nx + C$

Q.26 $\int \frac{1}{(1+x^2)\tan^{-1}x}dx$ = *(Faisalabad Board 2013)*
(D.G.K Board 2012 G-I)

(a) $\ln|1+x^2| + C$ (b) $(1+x^2)^2 + C$

(c) $\ln|\tan^{-1}x| + C$ (d) $\tan^{-1}x + C$

Q.27 If $\phi'(x) = f(x)$, then $\phi(x)$ is called of $f(x)$:
(Faisalabad Board 2013)

(a) Derivative
(b) Integral
(c) Differential co-efficient
(d) Area

Q.28 $\int a^x \ln a\,dx$ = *(D.G.K Board 2015 G-II)*

(a) $a^x + C$ (b) $\frac{a^x}{\ln a}+C$

(c) $\frac{a^x}{\ln a}+C$ (d) $\ln a\, a^x + C$

Q.29 $\int \text{Cosx } e^{\sin x}dx$ *(D.G.K Board 2015 G-II)*
(Gujranwala Board 2016)

(a) $e^{\cos x}$ (b) $\sin x\, e^{\cos x} + C$

(c) $e^{\sin x} + C$ (d) $e^{\sin x}\cos x + C$

Q.30 $\int e^{ax}dx$ *(D.G.K Board 2015 G-I)*

(a) e^{ax} (b) $a\,e^{ax}$

(c) $x\,e^{ax}$ (d) $\frac{e^{ax}}{a}$

Q.31 $\int \sec^2 x\tan x\,dx$ *(D.G.K Board 2015 G-I)*

(a) $\sec x\tan^2 x$ (b) $\frac{\sec^3 x}{3}$

(c) $\frac{\tan^2 x}{2}$ (d) $\frac{\sec^3 x\tan x}{3}$

Q.32 $\int \sec^2 x\, dx$ = *(D.G.K Board 2013 G-II)*

(a) $\cos x + C$ (b) $-\cos x + C$

(c) $\tan x + C$ (d) $\cot x + C$

Q.33 $\int \frac{\sec^2 x\, dx}{\tan x}$ *(D.G.K Board 2013 G-II)*

(a) $ln\,|\tan x| + C$ (b) $ln\,|\cot x| + C$

(c) $\cot x + C$ (d) $\tan x + C$

Q.34 $\int \frac{dx}{\sqrt{a^2 - x^2}} = ?$ *(D.G.K Board 2013 G-II)*

(a) $\sin^{-1}\frac{x}{a}$ (b) $\cos^{-1}\frac{x}{a}$

(c) $\tan^{-1}\frac{x}{a}$ (d) $-\sin^{-1}\frac{x}{a}$

Q.35 $\int e^{\cos x} \sin x\, dx$ = *(D.G.K Board 2013 G-II)*

(a) $-\sin x\, e^{\cos x} + C$ (b) $e^{\sin x} + C$

(c) $-e^{\cos x} + C$ (d) $\frac{e^{\sin x}}{\cos x} + C$

Q.36 $\int e^x\, dx$ equals: *(D.G.K Board 2013 G-II)*

(a) $e^x + C$ (b) $2e^x + C$

(c) $x\, e^{x-1} + C$ (d) $\frac{e^{x+1}}{x+1} + C$

Q.37 $\int e^{f(x)} f'(x)\, dx$ equals: *(D.G.K Board 2013 G-I)*

(a) $ln\,|f(x)| + C$ (b) $ln\,|f'(x)| + C$

(c) $e^{f'(x)} + C$ (d) $e^{f(x)} + C$

Q.38 $\int \cos(ax + b)\, dx$ = ---------- + c, $(a \neq 0)$

(a) $\frac{1}{a}\cos(ax + b)$ (b) $-\frac{1}{a}\cos(ax + b)$

(c) $\frac{1}{a}\sin(ax + b)$ (d) $-\frac{1}{a}\sin(ax + b)$

Q.39 $\int x\sqrt{x+2}\, dx$ =

(a) $\frac{2}{5}(x+2)^{3/2} - \frac{4}{3}(x+2)^{5/2} + c$

(b) $\frac{2}{3}(x+2)^{5/2} - \frac{2}{5}(x+2)^{3/2} + c$

(c) $\frac{2}{5}(x+2)^{5/2} + \frac{4}{3}(x+2)^{3/2} + c$

(d) $\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + c$

Q.40 $\int \text{cosec}\, x\, dx$ =

(a) $\ln|\sec x + \tan x| + c$

(b) $\ln|\text{cosec}\, x - \cot x| + c$

(c) $\ln|\sec x - \tan x| + c$

(d) $\ln|\text{cosec}\, x + \cot x| + c$

Q.41 $\int \sec(ax + b)\tan(ax + b)\, dx$ = ----------, $(a \neq 0)$

(a) $\frac{1}{a}\tan(ax + b) + c$ (b) $-\frac{1}{a}\tan(ax + b) + c$

(c) $\frac{1}{a}\sec(ax + b) + c$ (d) $-\frac{1}{a}\sec(ax + b) + c$

Q.42 $\int \cot(ax + b)\, dx$ = ---------- + c, $(a \neq 0)$

(a) $\frac{1}{a}\ln|\cos(ax + b)|$ (b) $-\frac{1}{a}\ln|\sin(ax + b)|$

(c) $\frac{1}{a}\ln|\sin(ax + b)|$ (d) $-\frac{1}{a}\ln|\cos(ax + b)|$

Q.43 $\int \text{cosec}^2(ax + b)\, dx$ = ----------, $(a \neq 0)$

(a) $-\frac{1}{a}\cot(ax + b) + c$

(b) $\cot(ax + b) + c$

(c) $\text{cosec}(ax + b) + c$

(d) $-\text{cosec}(ax + b)\cot(ax + b) + c$

Q.44 $\int \text{cosec}(ax + b)\cot(ax + b)\, dx$ = -------- + c, $(a \neq 0)$

(a) $-\text{cosec}(ax + b)$ (b) $-\frac{1}{a}\text{cosec}(ax + b)$

(c) $-\text{cosec}^2(ax + b)$ (d) $-\frac{1}{a}\cot(ax + b)$

Q.45 $\int \sec x \tan x\, dx$ =

(a) $\tan x + c$ (b) $-\tan x + c$

(c) $\sec x + c$ (d) $-\sec x + c$

Q.46 $\int \sin(ax + b)\, dx$ = ---------- + c, $(a \neq 0)$

(a) $\frac{1}{a}\cos(ax + b)$ (b) $-\frac{1}{a}\cos(ax + b)$

(c) $\frac{1}{a}\sin(ax + b)$ (d) $-\frac{1}{a}\sin(ax + b)$

Q.47 $\int \sec^2(ax + b)\, dx$ = ----------, $(a \neq 0)$

(a) $\frac{1}{a}\tan(ax + b) + c$

(b) $\tan(ax + b) + c$

(c) $\frac{1}{a}\sec(ax + b) + c$

(d) $-\frac{1}{a}\sec(ax + b)\tan(ax + b) + c$

Q.48 $\int \text{cosec}\, x \cot x\, dx$ =

(a) $\text{cosec}\, x + c$ (b) $-\text{cosec}\, x + c$

(c) $\cot x + c$ (d) $-\cot x + c$

Q.49 $\int \frac{1}{x\, ln\, x}\, dx$ = *(D.G.K Board 2015 G-I)*

(Multan Board 2017 G-I)(Lahore Board 2014 G-I)
(Lahore Board 2015 G-II)(Bahawalpur Board 2014)
(Sahiwal Board 2014)

(a) $ln\,(ln\, x)$ (b) x

(c) $\frac{ln\, x}{x}$ (d) $\frac{1}{x}$

Q.50 $\int \sec x\,dx =$ ---------- + c.
(Lahore Board 2014 G-II)
(a) $\ln|\sec x + \tan x|$ (b) $\ln|\operatorname{cosec} x - \cot x|$
(c) $\ln|\sec x - \tan x|$ (d) $\ln|\operatorname{cosec} x + \cot x|$

Q.51 $\int \sin^2 x\,dx =$
(a) $\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c$ (b) $\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$
(c) $\frac{1}{2}(x + \sin 2x) + c$ (d) $\frac{1}{2}(x - \sin 2x) + c$

Q.52 $\int \cos^2 x\,dx =$
(a) $\frac{1}{2}(x + \sin 2x) + c$ (b) $\frac{1}{2}(x - \sin 2x) + c$
(c) $\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c$ (d) $\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$

Q.53 $\int\left(\frac{x+2}{x+1}\right)dx =$
(a) $\ln|x+1| + c$ (b) $x + |x+1| + c$
(c) $x - \ln|x+1| + c$ (d) $x + \ln|x+1| + c$

Q.54 $\int\left(\frac{1-\sin x}{x+\cos x}\right)dx =$
(a) $\ln|1 - \sin x| + c$ (b) $\ln|1 + \sin x| + c$
(c) $\ln|x + \cos x| + c$ (d) $|x + \cos x| + c$

Q.55 $\int\frac{\sin(\ln x)}{x}\,dx =$
(a) $\cos(\ln x) + c$ (b) $-\sin(\ln x) + c$
(c) $\sin(\ln x) + c$ (d) $-\cos(\ln x) + c$

Q.56 If $\frac{1}{\sqrt{x^2-1}} = f'(x)$, then f(x).
(Sahiwal Board 2018)
(a) $\cos^{-1} x$ (b) $\sinh^{-1} x$
(c) $\cosh^{-1} x$ (d) $\operatorname{cosech}^{-1} x$

Q.57 $\int\sqrt{\frac{1-x}{1+x}}\,dx =$
(a) $\sin^{-1} x + \sqrt{1-x^2}$ (b) $\cos^{-1} x + \sqrt{1-x^2} + c$
(c) $\sin^{-1} x + \sqrt{x^2-1}$ (d) None of these

Q.58 $\int\left(\frac{\sec^2 x}{\tan x}\right)dx =$
(a) $\ln|\tan x| + c$ (b) $\frac{\sec^3 x}{3} + c$
(c) $\ln|\tan^2 x| + c$ (d) $\frac{\tan^2 x}{2} + c$

Q.59 $\int\frac{x^4-1}{x^2-1}\,dx =$
(a) $\frac{x^4}{4} - x + c$ (b) $\frac{x^4}{4} + x + c$
(c) $\frac{x^3}{3} - x + c$ (d) $\frac{x^3}{3} + x + c$

Q.60 $\int\frac{1}{ax+b}\,dx =$ ---------. *(Gujranwala Board 2014)*
(a) $\ln|ax+b| + c$ (b) $\frac{\ln|ax+b|}{b} + c$
(c) $\frac{(ax+b)^{-1+1}}{-1+1} + c$ (d) $\frac{\ln|ax+b|}{a} + c$

Q.61 $\int(2x+3)^{1/2}\,dx$ is equal to:
(Lahore Board 2014 G-I)
(a) $\frac{1}{2}(2x+3)^{1/2} + c$ (b) $\frac{2}{3}(2x+3)^{1/2} + c$
(c) $\frac{1}{3}(2x+3)^{1/2} + c$ (d) $\frac{1}{3}(2x+3)^{3/2} + c$

Q.62 If $\int_2^k 2x\,dx = 12$, then k =
(Gujranwala Board 2013)
(a) 2, –2 (b) 2, 6
(c) 4, –4 (d) 4, 2

Q.63 If the expression involves $\sqrt{x^2-a^2}$, then the suitable substitution is:
(Gujranwala Board 2013)
(a) x = a Sin θ (b) x = a Sec θ
(c) x = a Cos θ (d) x = Sin θ

Q.64 $\int\tan(ax+b)\,dx =$ *(Lahore Board 2013 G-II)*
(a) $\frac{1}{a}\operatorname{Cos}(ax+b) + c$
(b) $\frac{1}{a}\operatorname{Sin}(ax+b) + c$
(c) $\frac{1}{b}\ln|\operatorname{Sec}(ax+b)| + c$
(d) $\frac{1}{a}\ln|\operatorname{Sec}(ax+b)| + c$

Q.65 $\int\tan^{-1}\left(\sqrt{\frac{1-\operatorname{Cos}2x}{1+\operatorname{Cos}2x}}\right)dx =$
(Lahore Board 2013 G-II)
(a) $\frac{x^2}{2} + c$ (b) $x^2 + c$
(c) $2x^2 + c$ (d) $12x + c$

Q.66 If α is constant, then $\int\cot\alpha\,dy =$
(Lahore Board 2013 G-I)
(a) Sin x + c (b) $-\operatorname{Sin}\alpha + c$
(c) y Cot α + c (d) x Sin α + c

Q.67 Useful substitution of $\sqrt{a^2-x^2}\,dx$
(Gujranwala Board 2012)
(a) x = a Sinθ (b) x = Secθ
(c) x = a tanθ (d) x = Sinθ

Q.68 $\int\operatorname{Sin} h\,x\,dx =$ *(Gujranwala Board 2012)*
(a) $-\operatorname{Cos} hx + c$ (b) $\operatorname{Cos} hx + c$
(c) $\ln|\operatorname{Cos} hx| + c$ (d) $\operatorname{Cosec} hx + c$

Q.69 $\int$Cot (ax + b) dx = *(Lahore Board 2012 G-II)*

(a) a ln(Sin (ax + b)) + c

(b) $\frac{1}{a}$ ln (Cos(ax + b)) + c

(c) $\frac{1}{a}$ ln (Sin(ax + b)) + c

(d) a ln (Cos(ax + b)) + c

Q.70 Suitable substitution for expression involving $\sqrt{a^2 - x^2}$ to be integrated is: *(Lahore Board 2012 G-II)*

(a) x = a Secθ (b) x = a tanθ

(c) x = a Cosθ (d) x = a Sinθ

Q.71 $\int x\sqrt{1 + 2x^2}$ dx = *(Lahore Board 2012 G-I)*

(a) $\frac{1}{4}(1 + 2x^2)^{1/2} + c$ (b) $\frac{1}{6}(1 + 2x^2)^{3/2} + c$

(c) $\frac{1}{12}(1 + 2x^2)^{3/2} + c$ (d) $(1 + 2x^2)^{3/2} + c$

Q.72 $\int$Sin x dx is equal *(Lahore Board 2015 G-I & II)*

(a) Cos x (b) −Cos x

(c) Sin x (d) −Sin x

Q.73 $\int \frac{1}{1 + x^2}$ dx is equal to: *(Lahore Board 2015 G-II)*

(a) $\tan^{-1}x$ (b) $\tan^{-1}x^2$

(c) $\cot^{-1}x$ (d) $\cot^{-1}x^2$

Q.74 $\int e^{\tan x}$ $\sec^2$ x dx *(Multan Board 2017 G-I)* *(Rawalpindi Board 2018)*

(a) $-e^{\tan x} + C$ (b) $e^{\tan x} + C$

(c) $e^{\sec x} + C$ (d) $e^{\cos x} + C$

Q.75 $\int e^{\tan x}$ Sec^2x dx = *(Bahawalpur Board 2016)*

(a) $e^{\tan x}$ (b) $e^{-\tan x}$

(c) $e^{\cot x}$ (d) $e^{-\cot x}$

Q.76 $\int \text{Cot}^3$x (−Cosec^2x) dx = *(Bahawalpur Board 2016)*

(a) $\frac{\text{Cot}^3x}{3}$ (b) $-\frac{\text{Cot}^3x}{3}$

(c) $\frac{\text{Cot}^4x}{4}$ (d) $-\frac{\text{Cot}^4x}{4}$

Q.77 $\int\sqrt{2x + 3}$ (2 dx) *(Bahawalpur Board 2016, 2018)*

(a) $\frac{2}{3}(2x + 1)^{3/2}$ (b) $\frac{3}{2}(2x + 1)^{3/2}$

(c) $-\frac{2}{3}(2x + 1)^{3/2}$ (d) $-\frac{3}{2}(2x + 1)^{3/2}$

Q.78 $\int \frac{\text{Sec}^2x\ dx}{\tan x}$ *(Faisalabad Board 2016)*

(a) tan x (b) cot x

(c) ln cot x (d) ln tan x

Q.79 $\int \text{Cosec}^2 2x$ dx *(Faisalabad Board 2016)*

(a) Cos2x (b) $-\frac{1}{2}$ Cot 2x

(c) 2 tan2x (d) $\frac{1}{2}$Cot2x

Q.80 $\int \text{Sec}^2 2x$ dx = *(Gujranwala Board 2016)*

(a) $\frac{1}{2}$ tan 2x (b) tan 2x

(c) $\frac{1}{2}$ tanx (d) 2 tan 2x

Q.81 $\int \frac{\ln x}{x}$ dx *(Gujranwala Board 2016, 2018)*

(a) x (b) $\frac{(\ln x)^2}{2}$

(c) $\frac{1}{\ln x}$ (d) ln (ln x)

Q.82 $\int \tan\frac{\pi}{4}$ dx = *(Rawalpindi Board 2016)*

(a) ln |Sin $\frac{\pi}{4}$| (b) π

(c) $\text{Sec}\frac{\pi}{4}$ (d) x tan $\frac{\pi}{4}$

Q.83 $\int \frac{\text{Sin p}}{\text{Cos}^2x}$ dx = *(Rawalpindi Board 2016)*

(a) Sinp Sec^2x (b) Sinp tanx

(c) Cosp Sec^2x (d) Sec^2x

Q.84 $\int$Sec5x tan5x dx = *(Sargodha Board 2016)*

(a) $\frac{\text{Sec5x}}{5} + c$ (b) $\frac{\tan5x}{5} + c$

(c) 5 Sec5x + c (d) 5 tan5x + c

Q.85 $\int$(ln x) × $\frac{1}{x}$ dx *(Sargodha Board 2016)*

(a) ln x + c (b) $(\ln x)^2 + c$

(c) $\frac{(\ln x)^2}{2} + c$ (d) $-\frac{(\ln x)^2}{2} + c$

Q.86 To integrate $\int \frac{dx}{2\sqrt{4x^2 + 9}}$ dx we will make substitution.

(a) x = 3tanθ (b) 2x = 3tanθ

(c) x = tanθ (d) x = 9tanθ

Q.87 $\int 3^{\lambda x}$ dx = *(Sargodha Board 2017)*

(a) $\frac{3^{\lambda x}}{\lambda \ln 3} + C$ (b) $\frac{3^{\lambda x}}{\lambda} \ln 3 + C$

(c) $\frac{\lambda 3^{\lambda x}}{\ln 3} + C$ (d) $\frac{3^{\lambda x}}{\ln 3} + C$

Q.88 When expression $\sqrt{a^2 - x^2}$ involves in integration we substitute. *(Sargodha Board 2017)*

(a) x = a cosec θ (b) x = a tan θ

(c) x = a sec θ (d) x = a sin θ

Q.89 $\int \frac{dx}{a^2+x^2} =$ *(D.G.K Board 2010)*

(a) $\frac{1}{a}\cot^{-1}\frac{x}{a}$ (b) $\frac{1}{a}\tan^{-1}\frac{x}{a}$

(c) $-\frac{1}{a}\tan^{-1}\frac{x}{a}$ (d) $a\cot^{-1}\frac{x}{a}+C$

Q.90 $\int \frac{\cos(\ln x)}{x}dx =$

(a) $\cos(\ln x)+c$ (b) $-\cos(\ln x)+c$

(c) $-\sin(\ln x)+c$ (d) $\sin(\ln x)+c$

Q.91 $\int \frac{1}{x\ln x}dx =$

(a) $\ln x + c$ (b) $\ln(\ln(\ln x))+c$

(c) $\ln(\ln x)+c$ (d) $\ln x^2+c$

Q.92 $\int \frac{\cot\sqrt{x}\,dx}{\sqrt{x}}dx =$ *(Sargodha Board 2018)*

(a) $2\log_e|\sin\sqrt{x}|+C$ (b) $2\log_a|\sin\sqrt{x}|+C$

(c) $\log_e|\sin\sqrt{x}|+C$ (d) $\log_a|\sin\sqrt{x}|+C$

Q.93 $\int \frac{dx}{\sqrt{5-x^2}} =:$ *(Bahawalpur Board 2019)*

(a) $\text{Sin}^{-1}\frac{5}{x}$ (b) $\text{Sin}^{-1}\frac{x}{\sqrt{5}}$

(c) $\text{Sin}^{-1}\frac{x}{5}$ (d) $\text{Sin}^{-1}\frac{\sqrt{5}}{x}$

Q.94 $\frac{1}{\sqrt{x^2-1}}$ is derivative of:

(Faisalabad Board 2019)

(a) $\sinh^{-1}x$ (b) $\cosh^{-1}x$

(c) $\tanh^{-1}x$ (d) $\text{corth}^{-1}x$

Q.95 $\int (4-x^2)^{-\frac{1}{2}}(-2x)\,dx =:$

(Faisalabad Board 2019)

(a) $2\sqrt{4-x^2}$ (b) $\frac{1}{2}\sqrt{4-x^2}$

(c) $\ln(4-x^2)$ (d) $\ln\sqrt{4-x^2}$

Q.96 $\int \tan x\sec^2 x\,dx =:$ *(Bahawalpur Board 2019)*

(Faisalabad Board 2019 G-II)

(a) $\tan x + c$ (b) $\sec^2 x + c$

(c) $\sec x + c$ (d) $\frac{\tan^2 x}{2}+c$

Q.97 $\int e^{\tan x}\sec^2 x\,dx =$ *(Gujranwala Board 2019 G-I)*

(a) $-e^{\tan x}+c$ (b) $e^{\tan x}+c$

(c) $e^{\tan^2 x}+c$ (d) $e^{\cos x}+c$

Q.98 $\int \frac{e^x}{e^x+3}dx =:$ *(Lahore Board 2019)*

(a) $\ln(e^x+3)+c$ (b) $e^{2x}+c$

(c) e^0+c (d) $e^{2x}+3+c$

Q.99 $\int 5^{2x}\,dx =$ *(Multan Board 2019)*

(a) 5^{2x} (b) $2(5^{2x})/\ln 5$

(c) $5^{2x}\lambda n5$ (d) $2(5^{2x}\lambda n5)$

Q.100 $\int \frac{1}{1+x^2}e^{\tan^{-1}x}dx =$ *(Rawalpindi Board 2019)*

(a) $e^{\sec x}+c$ (b) $e^{\tan x}+c$

(c) $e^{-\tan x}+c$ (d) $e^{\tan^{-1}x}+c$

Q.101 $\int \frac{1}{x^2+2x+5}dx$ equals *(Sargodha Board 2019)*

(a) $2\tan^{-1}\left(\frac{x+1}{2}\right)+c$ (b) $2\tan^{-1}\left(\frac{x-1}{2}\right)+c$

(c) $\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c$ (d) $\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+c$

EXERCISE 3.4

SHORT ANSWERS TO THE QUESTIONS

Q.1 Evaluate $\int \ln x\,dx$ *(Faisalabad 2019 G-II)*

(Gujranwala Board 2010, 2011, 2012, 2014)
(Rawalpindi Board 2014)(D.G Khan Board 2017 G-I)

Sol: $\int \ln x\,dx = \int \underset{I}{\ln x}\cdot\underset{II}{1}\,dx$

Integrating by parts, we have

$= \lambda n\,x\,.\,x - \int x\times\frac{1}{x}dx = x\,\lambda n\,x - \int 1\,dx$

$= x\,\lambda n\,x - x + c$

Q.2 Evaluate $\int$ x. ln x dx *(Bahawalpur Board 2019)*

(Lahore Board 2015 G-II)(Gujranwala Board 2016)
(Lahore Board 2017 G-II)(Sargodha Board 2017)
(D.G Khan Board 2017 G-II)(Multan Board 2017 G-I)
(Sahiwal Board 2018, 2019)(Faisalabad Board 2019)

Sol: $\int \underset{I}{\ln x}\cdot\underset{II}{x}\,dx$ Integrating by parts, we have

$= \lambda n\,x\cdot\left(\frac{x^2}{2}\right) - \int\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)dx$

$= \frac{x^2}{2}\lambda n\,x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\lambda n\,x - \frac{1}{2}\left(\frac{x^2}{2}\right)+c$

$= \frac{1}{2}x^2\left(\ln x - \frac{1}{2}\right)+c$

Q.3 Evaluate $\int x^4 . \ln x \, dx$

Sol: $\int x^4 . \ln x \, dx = \int \ln x . x^4 \, dx$

Integrating by parts, we have

$$= \lambda n \, x \left(\frac{x^5}{5}\right) - \int \left(\frac{x^5}{5}\right)\left(\frac{1}{x}\right) dx$$

$$= \frac{x^5}{5} \lambda n \, x - \frac{1}{5} \int x^4 \, dx$$

$$= \frac{x^5}{5} \lambda n \, x - \frac{1}{5}\left(\frac{x^5}{5}\right) + c = \frac{x^5}{5}\left(\ln x - \frac{1}{5}\right) + c$$

Q.4 Find $\int Tan^{-1} x dx$. *(Sargodha Board 2018)*
(Faisalabad Board 2013)(Sargodha Board 2016)
(Multan Board 2018 G-II)(Lahore Board 2013)
(Gujranwala Board 2018, 2019)

Ans. $\int \tan^{-1} x \, dx$

$$= \int 1.\tan^{-1} x \, dx$$

Integrating by parts

$$= \tan^{-1}.x - \int x \frac{1}{1+x^2} dx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2} dx$$

$$= x\tan^{-1} x - \frac{1}{2} \ln |1 + x^2| + C$$

Q.5 Evaluate $\int x^2 \tan^{-1} x \, dx$

(Lahore Board 2014 G-I)

Sol: $\int \underset{I}{x^2} \underset{II}{\tan^{-1} x} \, dx = \int \tan^{-1} x \cdot x^2 dx$

$$= (\tan^{-1} x)\frac{x^3}{3} - \int \frac{x^3}{3} \times \frac{1}{1+x^2} dx$$

$$= \frac{x^3}{3}(\tan^{-1} x) - \frac{1}{3}\int \frac{x^3}{x^2+1} dx$$

$$= \frac{x^3}{3}(\tan^{-1} x) - \frac{1}{3}\int \left\{x - \frac{x}{x^2+1}\right\} dx$$

$$= \frac{x^3}{3}(\tan^{-1} x) - \frac{1}{3}\int x \, dx + \frac{1}{3}\int \frac{x}{x^2+1} dx$$

$$= \frac{x^3}{3}(\tan^{-1} x) - \frac{1}{3}\frac{x^2}{2} + \frac{1}{6}\int \frac{2x}{x^2+1} dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{x^2}{6} + \frac{1}{6}\lambda n \, (x^2 + 1) + c$$

Q.6 Evaluate $\int \sin^{-1} x \, dx$

(D.G Khan Board 2014 G-I)(Multan Board 2014 G-II)
(Gujranwala Board 2019 G-II)

Sol: $\int \sin^{-1} x \, dx = \int (\sin^{-1} x)(1) \, dx$

$$= (\sin^{-1} x)(x) - \int x \times \frac{1}{\sqrt{1-x^2}} dx$$

$$= x \sin^{-1} x + \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}} (-2x) dx$$

$$= x \sin^{-1} x + \frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + c$$

Q.7 Evaluate $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

(Lahore Board 2012 G-II)

Sol: $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$ .....(1)

Putting $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$ and $x = \sin t$

in (1), we have

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx = \int t . \sin t \, dt$$

Integrating by parts, we have

$$= t . (-\cos t) - \int (-\cos t)(1) \, dt$$

$$= -t \cos t + \int \cos t \, dt$$

$$= -t \cos t + \sin t + c = \sin t - t \cos t + c$$

$$= \sin(\sin^{-1} x) - \sin^{-1} x . \cos(\sin^{-1} x) + c$$

$$(\Theta \; t = \sin^{-1} x)$$

$$= x - \sin^{-1} x . \sqrt{\cos^2(\sin^{-1} x)} + c$$

$$= x - \sin^{-1} x . \sqrt{1 - \sin^2(\sin^{-1} x)} + c$$

$$= x - \sin^{-1} x . \sqrt{1 - [\sin(\sin^{-1} x)]^2} + c$$

$$= x - \sin^{-1} x . \sqrt{1 - x^2} + c$$

$$= x - \sqrt{1-x^2} . \sin^{-1} x + c$$

Q.8 Evaluate $\int (\ln x)^2\,dx$ *(Lahore Board 2010 G-I)*

Sol: $\int (\ln x)^2\,dx = \int \underset{I}{(\ln x)^2} \cdot \underset{II}{1}\,dx$

$\int (\ln x)^2 \cdot 1\,dx$

Integrating by parts, we have

$= (\lambda n\ x)^2 . (x) - \int (x) \times 2 \ln x \left(\frac{1}{x}\right) dx$

$= x\ (\lambda n\ x)^2 - 2 \int \ln x . 1\ dx$

Again integrating by parts, we have

$= x\ (\lambda n\ x)^2 - 2\left\{ \ln x . (x) - \int (x) \left(\frac{1}{x}\right) dx \right\}$

$= x\ (\lambda n\ x)^2 - 2x\ \lambda n\ x + 2 \int 1\ dx$

$= x\ (\lambda n\ x)^2 - 2\ x\ \lambda n\ x + 2x + c$

$= x\ \lambda n\ x\ (\lambda n\ x - 2) + 2\ x + c$

Q.9 Find $\int \sec^4 x dx$. *(Faisalabad Board 2013)*
(Bahawalpur Board 2019)

Ans. $\int \sec^4 x dx$.

$= \int \sec^2 x . \sec^2 x dx$

$= \int \sec^2 x (1 + \tan^2 x) dx \quad \because \sec^2 x = 1 + \tan^2 x$

$= \int \sec^2 x\,dx + \int \tan^2 x\ \sec^2 x\,dx$

$= \tan x + \frac{\tan^3 x}{3} + C$

Q.10 Evaluate $\int e^x (\cos x + \sin x)\,dx$
(Lahore Board 2014 G-I,II, 19)
(Gujranwala Board 2013, 19)
(Rawalpindi Board 2013)(Multan Board 2018 G-II)

Ans. $\int e^x (\cos x + \sin x)\,dx$

$= \int e^x \cos x\,dx + \int e^x \sin x\,dx$

$= e^x \sin - \int e^x \sin x\,dx + \int e^x \sin x\,dx$

$= e^x \sin x + c$

Q.11 Evaluate $\int e^x \left(\frac{1}{x} + \ln x\right) dx$ *(A.J.K Board 2017)*
(Rawalpindi Board 2016)(Bahawalpur Board 2016)
(Lahore Board 2018, 2019 G-II)
(Gujranwala Board 2019)(Sahiwal Board 2019)

Sol: $\int e^x \left(\frac{1}{x} + \ln x\right) dx = \int \ln x \cdot e^x\,dx + \int e^x \frac{1}{x}\,dx$

Integrating by parts first integral only, we have

$= (\lambda n\ x)\ e^x - \int e^x \frac{1}{x}\,dx + \int e^x . \frac{1}{x}\,dx = e^x\ \lambda n\ x + c$

Q.12 Evaluate $\int e^{ex} \left(a \sec^{-1} x + \frac{1}{x\sqrt{x^2 - 1}}\right)$
(Sargodha Board 2013)(Sahiwal Board 2019)

Ans. $\int e^{ex} \left(a \sec^{-1} x + \frac{1}{x\sqrt{x^2 - 1}}\right) dx$

Using $\int e^{ax} (af(x) + f'(x))dx = e^{ax} f(x) + c$

$= e^{ax} \sec^{-1} x + c$

Q.13 Evaluate $\int e^{3x} \left(\frac{3 \sin x - \cos x}{\sin^2 x}\right) dx$
(Rawalpindi Board 2014)(Lahore Board 2017 G-I)
(Lahore Board 2018 G-I)

Sol: $\int e^{3x} \left(\frac{3 \sin x - \cos x}{\sin^2 x}\right) dx$

$= 3 \int \frac{e^{3x} \sin x}{\sin^2 x} dx - \int \frac{e^{3x} \cos x}{\sin^2 x} dx$

$= 3 \int \frac{e^{3x}}{\sin x} dx - \int e^{3x} \frac{\cos x}{\sin x} \frac{1}{\sin x} dx$

$= 3 \int e^{3x} \operatorname{cosec} x\,dx - \int e^{3x} \operatorname{cosec} x \cot x\,dx$

Integrating first integral by parts, we have

$= 3 \left\{ \operatorname{coecex} \times \frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} \times (-\operatorname{cosec} x \cot x)\,dx \right\}$

$- \int e^{3x} \operatorname{cosec} x \cot x\,dx$

$= e^{3x} \operatorname{cosec} x + \int e^{3x} \operatorname{cosec} x \cot x\,dx$

$- \int e^{3x} \operatorname{cosec} x \cot x\,dx + c$

$= e^{3x} \operatorname{cosec} x + c$

Q.14 Evaluate $\int \frac{x\,e^x}{(1 + x)^2}\,dx$. *(Sahiwal Board 2018)*

Sol: $\int \frac{xe^x}{(1 + x)^2} dx = \int e^x \frac{x + 1 - 1}{(1 + x)^2} dx$

$= \int e^x \left(\frac{x + 1}{(1 + x)^2} - \frac{1}{(1 + x)^2}\right) dx$

$= \int e^x \left(\frac{1}{(1 + x)} - \frac{1}{(1 + x)^2}\right) dx$

Here $f(x) = \frac{1}{(1 + x)}$, $f'(x) = -\frac{1}{(1 + x)^2}$, $a = 1$

$\therefore \int e^x \left(\frac{1}{(1 + x)} - \frac{1}{(1 + x)^2}\right) dx = \int e^{ax} [a\ f(x) + f'(x)]\,dx$

$= e^{ax} f(x) + c = e^x \frac{1}{(1 + x)} + c$

LONG QUESTIONS

Q.1 Evaluate $\int x^4 \ln x \, dx$. *(Sahiwal Board 2015)*

Sol: See Short Question 9

Q.2 Evaluate $\int \tan^3 x \sec x \, dx$. *(Lahore Board 2017 G-I)*

Sol: See Short Question 21

Q.3 Evaluate $\int \sec^4 x \, dx$. *(D.G Khan Board 2017 G-II)*

Sol: See Short Question 22

Q.4 Evaluate $\int x^3 e^{5x} \, dx$ *(Gujranwala Boad 2018)*

Sol:
$$\int x^3 e^{5x} dx = x^3 \frac{e^{5x}}{5} - \frac{1}{5}\int 3x^2 e^{5x} dx$$
$$= x^3 \frac{e^{5x}}{5} - \frac{3}{5}\left[x^2 \frac{e^{5x}}{5} - \int 2x \frac{e^{5x}}{5} dx\right]$$
$$= x^3 \frac{e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6}{25}\int x e^{5x} dx$$
$$= x^3 \frac{e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6}{25}\left[x \cdot \frac{e^{5x}}{5} - \int 1 \cdot \frac{e^{5x}}{5} dx\right]$$
$$= x^3 \frac{e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6x e^{5x}}{125} - \frac{6}{125}\int e^{5x} dx$$
$$= x^3 \frac{e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6x e^{5x}}{125} - \frac{6}{125}\frac{e^{5x}}{5} + c$$
$$= \frac{e^{5x}}{5}\left(x^3 - \frac{3x^2}{5} + \frac{6x}{25} - \frac{6}{125}\right) + c$$

Q.5 Evaluate $\int x \sin^{-1} x \, dx$. *(Bahawalpur Board 2014)*

Sol:
$$\int (\sin^{-1} x)x \, dx$$
$$= \sin^{-1} x \int x \, dx - \int \left(\frac{d}{dx}(\sin^{-1} x) \int x \, dx\right) dx$$
$$= \sin^{-1} x \left(\frac{x^2}{2}\right) - \int \left(\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{x^2}{2}\right) dx$$
$$= \left(\frac{x^2}{2}\right) \sin^{-1} x + \frac{1}{2}\int \left(\frac{-x^2}{\sqrt{1-x^2}}\right) dx$$
$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$
$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2}\int \left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right) dx$$
$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2}\int \sqrt{1-x^2} \, dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}} dx$$
$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2}\left[\frac{1}{2}\sin^{-1} x + \frac{x}{2}\sqrt{1-x^2}\right] - \frac{1}{2}\sin^{-1} x + c$$

$$\left(\because \int \sqrt{(a^2 - x^2)} \, dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + c\right)$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{4}\sin^{-1} x - \frac{1}{2}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + c$$
$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + c$$

Q.6 Evaluate $\int \tan^3 x \sec x \, dx$

(Rawalpindi Board 2019)

Sol:
$$\int \tan^3 x \sec x \, dx = \int \tan^2 x \, . \, (\tan x \sec x) \, dx$$
$$= \int (\sec^2 x - 1)(\sec x \tan x) \, dx$$
$$= \int \sec^2 x (\sec x \tan x) \, dx - \int \sec x \tan x \, dx$$
$$= \int (\sec x)^2 (\sec x \tan x) \, dx - \sec x$$
$$= \frac{(\sec x)^3}{3} - \sec x + c$$
$$= \frac{1}{3}\sec^3 x - \sec x + c$$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 $\int x \sin x \, dx$ equals: *(Sahiwal Board 2014) (D.G.K Board 2017 G-I)*

(a) $x \sin x - \cos x + C$ (b) $-x \cos x + \sin x + C$
(c) $\sin x - \cos x + C$ (d) $\sin x + \cos x + C$

Q.2 $\int e^x (\cos x + \sin x) \, dx$ equals:
(Sahiwal Board 2014) (Sargodha Board 2016) (Multan Board 2013 G-I)(Lahore Board 2013 G-II)

(a) $e^x \cos x + C$ (b) $e^x \sin x + C$
(c) $e^x \tan x + C$ (d) $e^x \cot x + C$

Q.3 $\int e^{ax}(af(x) + f'(x))dx =$
(Faisalabad Board 2013, D.G.K Board 2014 G-I) (Rawalpindi Board 2017 G-I) (Gujranwala Board 2018)

(a) $e^x f(x) + C$ (b) $e^{ax} f'(x) + C$
(c) $e^{ax} f(x) + C$ (d) $\frac{e^{ax}}{f(x)} + C$

Q.4 $\int e^{-x} (\cos x - \sin x) \, dx =$
(Multan Board 2014, Rawalpindi Board 2013) (D.G.K Board 2017 G-I)(D.G.K Board 2011, 2017) (Lahore Board 2012 G-I)(Multan Board 2018 G-I)

(a) $e^{-x} \sin x + C$ (b) $e^{-x} \cos x + C$
(c) $e^x \sin x + C$ (d) $e^x \cos x + C$

Q.5 $\int \ln x \, dx =$ *(D.G.K Board 2012 G-I) (Rawalpindi Board 2017 G-II)(Sahowal Board 2017) (Lahore Board 2012 G-II)*

(a) $x \ln e + C$ (b) $x \ln x/e + C$
(c) $x \ln e/x + C$ (d) $x \ln x + C$

Q.6 $\int e^x \left(\ln x + \frac{1}{x}\right) dx$ *(Lahore Board 2017 G-I) (D.G.K Board 2012 G-I) (Lahore Board 2014 G-I) (Multan Board 2018 G-II)*

(a) $\ln x \, e^x + C$ (b) $\frac{e^x}{x} + C$
(c) $x \ln x + C$ (d) $\frac{\ln x}{x} + C$

Q.7 $\int \frac{\sin(\sec^{-1} x)}{x\sqrt{x^2-1}} dx =$ *(D.G.K Board 2012 G-I)*

(a) $\frac{\cos(\sec^{-1} x)}{x\sqrt{x^2-1}} + C$ (b) $-\cos(\sec^{-1} x) + C$
(c) $-\frac{\sin(\sec^{-1} x)}{x\sqrt{x^2-1}}$ (d) $\cos(\sec^{-1} x) + C$

Q.8 $\int \text{cosec}^2 x \, dx =$ ---------- + c.

(a) cot x (b) −cot x
(c) cosec x cot x (d) −cosec x cot x

Q.9 $\int (a \cos x + b \sin x) dx =$

(a) $a \sin x + b \cos x + c$ (b) $a \cos x + b \sin x + c$
(c) $a \cos x - b \sin x + c$ (d) $a \sin x - b \cos x + c$

Q.10 $\int e^{2x} (-\sin x + 2 \cos x) dx$ equal. *(Lahore Board 2015 G-II)*

(a) $e^{2x} \sin x$ (b) $e^{2x} \cos x$
(c) $-e^{2x} \sin x$ (d) $-e^{2x} \cos x$

Q.11 $\int e^x \frac{(x^2+1)}{(x+1)^2} dx =$

(a) $\left(\frac{x-1}{x+1}\right) e^x + c$ (b) $\left(\frac{x+1}{x-1}\right) e^x + c$
(c) $(x-1)e^x + c$ (d) None of these

Q.12 $\int \left(\frac{\sin x \cos x - 1}{\sin^2 x}\right) e^x dx =$

(a) $e^x \cot x + c$ (b) $e^x \tan x + c$
(c) $e^x \sec^2 x + c$ (d) None of these

Q.13 $\int \left(\frac{1+\sin x}{1+\cos x}\right) e^x dx =$

(a) $e^x \tan(x/2) + c$ (b) $e^x \cot(x/2) + c$
(c) $ex (\tan(x/2) + \cot(x/2)) + c$
(d) None of these

Q.14 $\int \text{Sec}^4 x \, dx =$ *(Lahore Board 2013 G-II)*

(a) $\frac{1}{3}\tan^3 x - \tan x + c$ (b) $\frac{1}{3}\tan^3 x + \tan x + c$
(c) $\frac{1}{3}\sec^3 x + c$ (d) $\frac{1}{3}\sec^3 x + \tan x + c$

Q.15 $\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx =$ *(Bahawalpur Board 2016)*

(a) $\frac{e^x}{x}$ (b) $x e^x$
(c) $\frac{e^x}{x^2}$ (d) $-\frac{e^x}{x^2}$

Q.16 $\int e^x \left(\frac{2 + \sin 2x}{2 \cos^2 x}\right) dx =$

(a) $e^x \tan x + c$ (b) $e^x \tan x + c$
(c) $-e^x \cot x + c$ (d) None of these

Q.17 $\int e^x \left[\frac{1}{1+x^2} + \tan^{-1} x\right] dx =$ *(Lahore Board 2017 G-II)*

(a) $e^x \tan x + C$ (b) $\frac{e^x}{1+x^2} + C$
(c) $e^x \sin x + C$ (d) $e^x \tan^{-1} x + C$

Q.18 $\int \lambda nx \, dx =$: *(Faisalabad Board 2019)*

(a) $\frac{1}{x}$ (b) $\frac{(\lambda nx)^2}{2}$
(c) $x\lambda nx$ (d) $x \lambda n x - x + c$

Q.19 $\int e^x (\cos x + \sin x) dx =$:

(Faisalabad Board 2019)

(a) $e^x \cos x$ (b) $e^x \sin x$
(c) $e^x \tan x$ (d) $\lambda n(\sin x)$

Q.20 $\int e^x \left[\sinh^{-1} x + \frac{1}{\sqrt{1+x^2}}\right] dx =$:

(Faisalabad Board 2019 G-II)

(a) $e^x \cosh^{-1} x$ (b) $e^x \cos^{-1} x$
(c) $e^x \sinh^{-1} x$ (d) $e^x \sin^{-1} x$

Q.21 $\int \ln x dx =$ *(Sahiwal Board 2019)*

(a) $x \ln x - x$ (b) $x \ln x + x$
(c) $x - x \ln x$ (d) $-x \ln x - x$

EXERCISE 3.5

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find $\int \frac{5x+8}{(x+3)(2x-1)} dx$.

(Gujranwala Board 2013)(Multan Board 2018 G-I)

Ans. $\int \frac{5x+8}{(x+3)(2x-1)} dx$

By partial fraction.

$$\frac{5x+8}{(x+3)(2x-1)} = \frac{A}{(x+3)} + \frac{B}{2x-1} \quad \text{..... (i)}$$

$5x + 8 = A(2x - 1) + B(x + 3)$ ..... (ii)

Put, $x + 3 = 0$ as $x = -3$ in (ii)

$5(-3) + 8 = A(2(-3) - 1)$

$-15 + 8 = A(-7) \Rightarrow -7A = -7$

$\boxed{A = 1}$

Put, $2x - 1 = 0$ as $x = \frac{1}{2}$ in ..... (ii)

$5\left(\frac{1}{2}\right) + 8 = B\left(\frac{1}{2} + 3\right)$

$\frac{5}{2} + 8 = B\left(\frac{7}{2}\right)$

$\frac{5 + 16}{2} = \frac{7}{2}B \Rightarrow \frac{21}{2} = \frac{7}{2}$ $\boxed{B = \frac{21}{7} = 3}$

Put value of 'A' and 'B' in (i).

$$\frac{5x + 8}{(x + 3)(2x - 1)} = \frac{1}{x + 3} + \frac{3}{2x - 1}$$

$$\int \frac{5x + 8}{(x + 3)(2x - 1)} = \int \frac{1}{x + 3}\,dx + 3\int \frac{1}{2x - 1}\,dx$$

$$\int \frac{5x + 8}{(x + 3)(2x - 1)} = ln|x + 3| + \frac{3}{2}\int \frac{1}{2x - 1}\,dx$$

$$\int \frac{5x + 8}{(x + 3)(2x - 1)} = ln|x + 3| + \frac{3}{2}\,ln|2x - 1| + c$$

Q.2 Evaluate $\int \frac{3 - x}{1 - x - 6x^2}\,dx$

(Sargodha Board 2013)(Lahore Board 2015 G-II)

Ans. $\int \frac{3 - x}{1 - x - 6x^2}\,dx$ .....(i)

Now, $1 - x - 6x^2 = 1 - 3x + 2x - 6x^2$

$= 1(1 - 3x) + 2x(1 - 3x) = (1 + 2x)(1 - 3x)$

So, equation (i) becomes,

$$\frac{3 - x}{1 - x - 6x^2} = \frac{A}{1 + 2x} + \frac{B}{1 - 3x}$$

$(3 - x) = A(1 - 3x) + B(1 + 2x)$

Put $(1 - 3x) = 0$ or $3x = 1$ or $x = \frac{1}{3}$

$3 - \frac{1}{3} = A(0) + B(1 + 2\left(\frac{1}{3}\right))$

$\frac{9 - 1}{3} = + B\left(\frac{3 + 2}{3}\right)$

$\frac{8}{3} = B\left(\frac{5}{3}\right) \Rightarrow$ $\boxed{B = \frac{8}{5}}$

Put $1 + 2x = 0$ or $2x = -1$ or $x = -\frac{1}{2}$

$3 - \left(-\frac{1}{2}\right) = A\left(1 - 3\left(-\frac{1}{2}\right)\right) + B(0)$

$3 + \frac{1}{2} = A\left(A + \frac{3}{2}\right)$

$\frac{6 + 1}{2} = A\left(\frac{2 + 3}{2}\right)$

$\frac{7}{3} = A\left(\frac{5}{3}\right) \Rightarrow$ $\boxed{A = \frac{7}{5}}$

$$\frac{3 - x}{1 - x - 6x^2} = \frac{\frac{7}{5}}{1 + 2x} + \frac{\frac{8}{5}}{1 - 3x}$$

$$= \frac{7}{5(1 + 2x)} + \frac{8}{5(1 - 3x)}$$

$$\int \frac{3 - x}{1 - x - 6x^2} = \frac{7}{5}\int \frac{dx}{1 + 2x} + \frac{8}{5}\int \frac{dx}{1 - 3x}$$

$$= \frac{7}{10}\int \frac{2dx}{1 + 2x} - \frac{8}{15}\int \frac{-3dx}{1 - 3x}$$

$$= \frac{7}{10}\,ln|1 + 2x| - \frac{8}{15}\,ln|1 - 3x| + c$$

Q.3 Evaluate $\int \frac{3x + 1}{x^2 - x - 6}dx$

(Multan Board 2013 G-I)(D.G.K Board 2013 G-II) (Sargodha Board 2017)

Ans. $\int \frac{3x + 1}{x^2 - 2 - 6}\,dx$ $\rightarrow$ (A)

Consider

$$\frac{3x + 1}{x^2 - x - 6} = \frac{3x + 1}{x^2 - 3x + 2x - 6} = \frac{3x + 1}{x(x-3) + 2(x-3)}$$

$$\frac{3x + 1}{x^2 - x - 6} = \frac{3x + 1}{(x - 3)(x + 2)}$$

Now we use partial fraction.

Let $\frac{3x + 1}{(x - 3)(x + 2)} = \frac{A}{x - 3} + \frac{B}{x + 2}$ $\rightarrow$ (1)

Multiplying (1) by $(x-3)(x+2)$ we get

$3x + 1 = A(x + 2) + B(x - 3)$ $\rightarrow$ (2)

Put $x - 3 = 0 \Rightarrow x = 3$ in (2)

$3(3) + 1 = A(3 + 2) + B(3 - 3)$

$10 = 5A + 0$

Or $A = \frac{10}{5} = 2$

$A = 2$

Put $x + 2 = 0 \Rightarrow x = -2$ in (2)

$3(-2) + 1 = A(-2 + 2) + B(-2 - 3)$

$-6 + 1 = 0 - 5B$

$-5 = -5\,B$

Now putting the value of A and B in (1)

$$\frac{3x+1}{(x-3)(x+2)} = \frac{2}{x-3} + \frac{1}{x+2}$$

Hence A becomes

$$\int \frac{3x+1}{(x-3)(x+2)}\,dx = \int\left(\frac{2}{x-3} + \frac{1}{x+2}\right)dx$$

$$= 2\int \frac{1}{x-3}\,dx + \int \frac{1}{x+2}\,dx = 2\ln|x-3| + \ln|x+2|$$

Q.4 **Find** $\int \frac{(a-b)x}{(x-a)(x-b)}\,dx$

(Faisalabad Board 2013)(Gujranwala Board 2012)

Ans. $\int \frac{(a-b)x}{(x-a)(x-b)}\,dx$

First by partial fraction

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \quad \text{..... (i)}$$

$(a-b)x = A(x-b) + B(x-a)$ ..... (ii)

Put $x - a = 0$ or $x = a$ in (ii)

$(a-b)a = A(a-b) \Rightarrow \boxed{A = a}$

Put $x - b = 0$ or $x = b$ in (ii)

$(a-b)b = B(b-a)$

$\frac{-(b-a)b}{(b-a)} = B \Rightarrow \boxed{B = -b}$

Put value of A and B in (i)

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{a}{x-a} + \frac{(-b)}{x-b}$$

By integrating

$$\int \frac{(a-b)x}{(x-a)(x-b)} = a\int \frac{1}{x-a}\,dx - b\int \frac{1}{x-b}\,dx$$

$$= a\ln|x-a| - b\ln|x-b| + C$$

Q.5 **Evaluate** $\int \frac{2x}{x^2 - a^2}\,dx$

(Lahore Board 2014 G-I, 2014 G-II)

Sol: $\int \frac{2x}{x^2-a^2}\,dx = \int \frac{2x}{(x-a)(x+a)}\,dx$

$$= \int \frac{(x+a)+(x-a)}{(x-a)(x+a)}\,dx$$

$$= \int\left[\frac{(x+a)}{(x-a)(x+a)} + \frac{(x-a)}{(x-a)(x+a)}\right]dx$$

$$= \int\left[\frac{1}{x-a} + \frac{1}{x+a}\right]dx$$

$$= \int \frac{1}{x-a}\,dx + \int \frac{1}{x+a}\,dx$$

$$= \lambda n|x-a| + \lambda n|x+a| + c = \lambda n|x-a||x+a| + c$$

$$= \lambda n|x^2 - a^2| + c$$

LONG QUESTIONS

Q.1 Evaluate $\int \frac{5x+8}{(x+3)(2x-1)}\,dx$.

(Rawalpindi Board 2016)

Sol: See Short Question 1

Q.2 Evaluate $\int \frac{3-x}{1-x-6x^2}\,dx$.

(D.G.K Board 2017 G-II)(Faisalabad Board 2017)

Sol: See Short Question 4

Q.3 Evaluate $\int \frac{2x^2}{(x-1)^2(x+1)}\,dx$.

(Multan Board 2014 G-I)(Bahawalpur board 2019)

Sol: Given that: $\int \frac{2x^2}{(x-1)^2(x+1)}\,dx$

Consider:

$$\frac{2x^2}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)} \quad \text{..... (1)}$$

Multiply both sides by $(x-1)^2(x+1)$:

$2x^2 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$ ..... (2)

$2x^2 = A(x^2 - 1) + B(x+1) + C(x^2 + 1 - 2x)$

$2x^2 = Ax^2 - A + Bx + B + Cx^2 + C - 2Cx$

$2x^2 = (A+C)x^2 + (B-2C)x + (-A+B+C)$ ..... (3)

Put $x + 1 = 0$, $x = -1$:

$2(-1)^2 = C(-1-1)^2$

$2(1) = C(-2)^2$

$2 = C(4) \Rightarrow \boxed{C = \frac{1}{2}}$

Put $x - 1 = 0$, $x = 1$:

$2(1)^2 = B(1+1)$

$2(1) = B(2)$

$2 = 2B \Rightarrow \boxed{B = 1}$

Comparing coefficient of x^2, in eq. (3):

$A + C = 2 \Rightarrow A + \frac{1}{2} = 2$

$A = 2 - \frac{1}{2} \Rightarrow A = \frac{4-1}{2} \Rightarrow \boxed{A = \frac{3}{2}}$

So, from eq. (1):

$$\frac{2x^2}{(x-1)^2(x+1)} = \frac{3}{2(x-1)} + \frac{1}{(x-1)^2} + \frac{1}{2(x+1)}$$

$$\int \frac{2x^2}{(x-1)^2(x+1)}\,dx$$

$$= \frac{3}{2}\int \frac{1}{(x-1)}\,dx + \int \frac{1}{(x-1)^2}\,dx + \frac{1}{2}\int \frac{1}{(x+1)}\,dx$$

$$= \frac{3}{2}\ln|x-1| + \int (x-1)^{-2}\,dx + \frac{1}{2}\ln|x+1|$$

$$= \frac{3}{2}\ln|x-1| + \frac{(x-1)^{-1}}{-1} + \ln|x+1|$$

$$= \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{(x-1)} + c$$

Q.4 Evaluate $\int \frac{7x-1}{(x-1)^2(x+1)}\,dx$.

(Sargodha Board 2016)

Sol: Given that: $\int \frac{7x-1}{(x-1)^2(x+1)}\,dx$

Consider:

$$\frac{7x-1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \quad \ldots\ldots (1)$$

Multiplying by $(x-1)^2(x+1)$

$7x-1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \ldots (2)$

$7x-1 = A(x^2-1) + B(x+1) + C(x^2-2x+1)$

$7x-1 = Ax^2 - A + Bx + B + Cx^2 - 2Cx + C$

$7x-1 = Ax^2 + Cx^2 + Bx - 2Cx + B - A + C$

$7x-1 = (A+C)x^2 + (B-2C)x + (B-A+C) \ldots\ldots (3)$

Put $x-1=0$, $x=1$:

$7(1)-1 = B(1+1)$

$7-1 = B(2) \Rightarrow 6 = 2B$

$B = \frac{6}{2} \Rightarrow \boxed{B = 3}$

Put $x+1=0$, $x=-1$:

$7(-1)-1 = C(-1-1)^2$

$-7-1 = C(-2)^2$

$-8 = C(4)$

$C = \frac{-8}{4} \Rightarrow \boxed{C = -2}$

Comparing coefficients of x^2, x, x^0:

x^2: $A + C = 0 \Rightarrow A - 2 = 0 \Rightarrow \boxed{A = 2}$

So, from eq. (1):

$$\frac{7x-1}{(x-1)^2(x+1)} = \frac{2}{x-1} + \frac{3}{(x-1)^2} + \frac{-2}{x+1}$$

$$= \frac{2}{x-1} + \frac{3}{(x-1)^2} - \frac{2}{x+1}$$

$$\int \frac{7x-1}{(x-1)^2(x+1)}\,dx$$

$$= \int \frac{2}{x-1}\,dx + \int \frac{3}{(x-1)^2}\,dx - \int \frac{2}{x+1}\,dx$$

$$= 2\int \frac{1}{x-1}\,dx + 3\int (x-1)^{-2}\,dx - 2\int \frac{1}{x+1}\,dx$$

$$= 2\ln|x-1| + 3\frac{(x-1)^{-1}}{-1} - 2\ln|x+1| + c$$

$$= \boxed{2\ln|x-1| - \frac{3}{x-1} - 2\ln|x+1| + c}$$

Q.5 Evaluate $\int \frac{1}{x(x^3-1)}\,dx$.

(Lahore Board 2015 G-II)

Sol: Given that:

$$\int \frac{3}{x(x^3-1)}\,dx = \int \frac{3}{x(x-1)(x^2+x+1)}\,dx$$

Consider:

$$\frac{3}{x(x-1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} \quad \ldots\ldots (1)$$

Multiplying by $x(x-1)(x^2+x+1)$

$3 = A(x-1)(x^2+x+1) + Bx(x^2+x+1) + (Cx+D)x(x-1) \ldots\ldots (2)$

$3 = A(x^3-1) + B(x^3+x^2+x) + (Cx+D)(x^2-x)$

$3 = Ax^3 - A + Bx^3 + Bx^2 + Bx + Cx^3 - Cx^2 + Dx^2 - Dx$

$3 = Ax^3 + Bx^3 + Cx^3 + Bx^2 - Cx^2 + Dx^2 + Bx - Dx - A$

$3 = (A+B+C)x^3 + (B-C+D)x^2 + (B-D)x - A \quad \ldots\ldots (3)$

Put $x = 0$:

$3 = A(0-1)((0)^2+0+1)$

$3 = A(-1)(0+0+1)$

$3 = A(-1)(1)$

$3 = -A \Rightarrow \boxed{A = -3}$

Put $x-1=0$, $x=1$:

$3 = B(1)((1)^2+1+1)$

$3 = B(1+1+1)$

$3 = B(3) \Rightarrow \boxed{B = 1}$

Comparing coefficients of x^3, x^2, x, x^0:

x^3: $A + B + C = 0 \Rightarrow -3 + 1 + C = 0$

$-2 + C = 0 \Rightarrow \boxed{C = 2}$

x^2: $B - C + D = 0 \Rightarrow 1 - 2 + D = 0$

$-1 + D = 0 \Rightarrow \boxed{D = 1}$

So, from eq. (1)

$$\frac{3}{x(x-1)(x^2+x+1)} = \frac{-3}{x} + \frac{1}{x-1} + \frac{2x+1}{x^2+x+1}$$

$$\int \frac{3}{x(x-1)(x^2+x+1)}\,dx$$

$$= \int \frac{-3}{x}\,dx + \int \frac{1}{x-1}\,dx + \int \frac{2x+1}{x^2+x+1}\,dx$$

$$= -3\int \frac{1}{x}\,dx + \int \frac{1}{x-1}\,dx + \int \frac{2x+1}{x^2+x+1}\,dx$$

$$= \boxed{-3\ln|x| + \ln|x-1| + \ln|x^2+x+1| + c}$$

Q.6 Evaluate $\int \frac{x^2+3x-34}{x^2+2x-15}\,dx$.

Sol: Given that:

$x^2+2x-15 \;)\; x^2+3x-34 \;(1$

$+x^2+2x-15$

$-\quad -\quad +$

$x-19$

$$\int \frac{x^2+3x-34}{x^2+2x-15}\,dx = \int\left(1-\frac{x-19}{x^2+2x-15}\right)dx$$

$$= x+\int \frac{x-19}{x^2+5x-3x-15}\,dx$$

$$= x+\int \frac{x-19}{x(x+5)-3(x+5)}\,dx$$

$$= x+\int \frac{x-19}{(x+5)(x-3)}\,dx \qquad \ldots (i)$$

Consider: $\frac{x-19}{(x+5)(x-3)} = \frac{A}{(x+5)}+\frac{B}{(x-3)}$ ..... (1)

Multiplying both sides by $(x+5)(x-3)$:

$x-19 = A(x-3)+B(x+5)$ ..... (2)

Put $x-3=0$, $x=3$:

$3-19 = B(3+5)$

$-16 = B(8) \Rightarrow \boxed{B=-2}$

Put $x+5=0$, $x=-5$:

$-5-19 = A(-5-3)$

$-24 = A(-8)$

$\frac{-24}{-8} = A \Rightarrow \boxed{A=3}$

So, from eq. (1):

$$\frac{x+9}{(x+5)(x-3)} = \frac{3}{x+5}-\frac{2}{x-3}$$

$$\int \frac{x-19}{(x+5)(x-3)}\,dx = 3\int \frac{1}{(x+5)}\,dx - 2\int \frac{1}{x-3}\,dx$$

$$\int \frac{x-19}{(x+5)(x-3)}\,dx = 3\ln|x+5| - 2\ln|x-3| + c$$

Now, from eq. (i):

$$\int \frac{x^2+3x-34}{x^2+2x-15}\,dx = x+3\ln|x+5| - 2\ln|x-3| + c$$

Q.7 Evaluate $\int \frac{2x^3-3x^2-x-7}{2x^2-3x-2}\,dx$

Sol: Given that: $\int \frac{2x^3-3x^2-x-7}{2x^2-3x-2}\,dx$

$2x^2-3x-2 \;)\; 2x^3-3x^2-x-7 \;(x$

$+2x^3-3x^2-2x$

$-\quad +\quad +$

$x-7$

$$= \int \left(x+\frac{x-7}{2x^2-3x-2}\right)dx$$

$$= \int x\,dx + \int \frac{x-7}{(2x^2-3x-2)}\,dx$$

$$= \frac{x^2}{2}+\int \frac{x-7}{2x^2-3x-2}\,dx = \frac{x^2}{2}+\int \frac{x-7}{2x^2-4x+x-2}\,dx$$

$$= \frac{x^2}{2}+\int \frac{x-7}{2x(x-2)+1(x-2)}\,dx$$

$$= \frac{x^2}{2}+\int \frac{x-7}{(x-2)(2x+1)}\,dx \qquad \ldots\ldots (i)$$

$$\frac{x-7}{(x-2)(2x+1)} = \frac{A}{(x-2)}+\frac{B}{(2x+1)} \qquad \ldots\ldots (1)$$

Multiplying both sides by $(x-2)(2x+1)$:

$x-7 = A(2x+1)+B(x-2)$ ..... (2)

Put $2x+1=0$, $2x=-1$, $x=-\frac{1}{2}$:

$-\frac{1}{2}-7 = B\left(-\frac{1}{2}-2\right)$

$\frac{-1-14}{2} = B\left(\frac{-1-4}{2}\right)$

$\Rightarrow -\frac{15}{2} = B\left(\frac{-5}{2}\right)$

$B = \frac{-15}{2}\times\frac{2}{-5} \Rightarrow \boxed{B=3}$

Put $x-2=0$, $x=2$:

$2-7 = A(2(2)+1)$

$-5 = A(4+1)$

$\Rightarrow -5 = A(5)$

$A = -\frac{5}{5} = -1 \Rightarrow \boxed{A=-1}$

So from eq. (1):

$$\frac{x-7}{(x-2)(2x+1)} = \frac{-1}{(x-2)}+\frac{3}{2x+1}$$

$$\int \frac{x-4}{(x-2)(2x+1)}\,dx = \int \frac{-1}{(x-2)}\,dx + \int \frac{3}{2x+1}\,dx$$

$$= -\ln|x-2| + \frac{3}{2}\int \frac{1}{2x+1}\,.\,2\,dx$$

$$= -\ln|x-2| + \frac{3}{2}\ln|2x+1| + c$$

Now, from eq. (i):

$$\int \frac{2x^3-3x^2-x-7}{2x^2-3x-2} = \frac{x^2}{2} - \ln|x-2| + \frac{3}{2}\ln|2x+1| + c$$

Q.8 Evaluate $\int \frac{1}{6x^2+5x-4}\,dx$

Sol: Given that:

$$\int \frac{1}{6x^2+5x-4} = \int \frac{1}{6x^2+8x-3x-4}\,dx$$

$$= \int \frac{1}{2x(3x+4)-1(3x+4)}\,dx = \int \frac{1}{(3x+4)(2x-1)}\,dx$$

Suppose:

$$\frac{1}{(3x+4)(2x-1)} = \frac{A}{(3x+4)} + \frac{B}{(2x-1)} \quad \ldots\ldots (1)$$

Multiplying by $(3x+4)(2x-1)$:

$$1 = A(2x-1) + B(3x+4) \quad \ldots\ldots (2)$$

Put $3x+4=0$, $3x=-4$, $x=-\frac{4}{3}$

$$1 = A\left[2\left(-\frac{4}{3}\right)-1\right]$$

$$1 = A\left(-\frac{8}{3}-1\right)$$

$$\Rightarrow 1 = A\left(\frac{-8-3}{3}\right)$$

$$1 = A\left(\frac{-11}{3}\right) \Rightarrow \boxed{A = \frac{-3}{11}}$$

Put $2x-1=0$, $2x=1$, $x=\frac{1}{2}$

$$1 = B\left[3\left(\frac{1}{2}\right)+4\right]$$

$$1 = B\left(\frac{3}{2}+4\right)$$

$$\Rightarrow 1 = B\left(\frac{3+8}{2}\right)$$

$$1 = B\left(\frac{11}{2}\right) \Rightarrow \boxed{B = \frac{2}{11}}$$

So, from eq. (1)

$$\frac{1}{(3x+4)(2x-1)} = \frac{-3}{11(3x+4)} + \frac{2}{11(2x-1)}$$

$$\int \frac{1}{(3x+4)(2x-1)}dx = \int \frac{-3}{11(3x+4)}dx + \int \frac{2}{11(2x-1)}dx$$

$$= -\frac{1}{11}\int \frac{3}{3x+4}dx + \frac{1}{11}\int \frac{2}{2x-1}dx$$

$$= -\frac{1}{11}\ln|3x+4| + \frac{1}{11}\ln|2x-1| + c$$

$$\int \frac{1}{6x^2+5x-4}dx = \frac{1}{11}\ln|2x-1| - \frac{1}{11}\ln|3x+4| + c$$

$$= \frac{1}{11}\ln\left|\frac{2x-1}{3x+4}\right| + c$$

Q.9 Evaluate $\int \frac{(a-b)x}{(x-a)(x-b)}dx$

(Gujranwala Board 2019)

Sol: See Short Question 7

Q.10 Integrate $\int \frac{12}{x^3+8}dx$

(Lahore Board 2019 G-II)

Sol: Given that:

$$\int \frac{12}{x^3+8}dx = \int \frac{12}{(x)^3+(2)^3}dx = \int \frac{12}{(x+2)(x^2+4-2x)}$$

Consider:

$$\frac{12}{(x+2)(x^2+4-2x)} = \frac{A}{(x+2)} + \frac{Bx+C}{(x^2+4-2x)} \quad \ldots (1)$$

Multiplying both sides by $(x+2)(x^2-2x+4)$:

$$12 = A(x^2+4-2x) + (Bx+C)(x+2) \quad \ldots (2)$$

$$12 = Ax^2 + 4A - 2Ax + Bx^2 + 2Bx + Cx + 2C$$

$$12 = (A+B)x^2 + (-2A+2B+C)x + (4A+2C) \quad \ldots (3)$$

Put $x+2=0$, $x=-2$

$$12 = A[(-2)^2 + 4 - 2(-2)]$$

$$12 = A(4+4+4)$$

$$12 = 12A \Rightarrow \boxed{A = 1}$$

Comparing coefficients of x^2, x^0 in eq. (3):

x^2: $A+B=0 \Rightarrow 1+B=0$

$\Rightarrow \boxed{B = -1}$

x^0: $4A+2C = 12 \Rightarrow 4(1)+2C = 12$

$4+2C = 12 \Rightarrow 2C = 12-4$

$2C = 8 \Rightarrow \boxed{C = 4}$

So, from eq. (1):

$$\frac{12}{(x+2)(x^2+4-x)} = \frac{1}{(x+2)} + \frac{-x+4}{(x^2+4-2x)}$$

$$= \frac{1}{x+2} + \frac{-1(x-4)}{x^2+4-2x} = \frac{1}{x+2} - \frac{x-4}{x^2+4-2x}$$

$$\int \frac{12}{(x+2)(x^2+4-x)}dx = \int \frac{1}{(x+2)}dx - \int \frac{(x-4)}{(x^2+4-2x)}dx$$

$$= \ln|x+2| - \frac{1}{2}\int \frac{2x-8}{x^2-2x+4}dx$$

$$= \ln|x+2| - \frac{1}{2}\int \frac{2x-2-6}{x^2+2x+4}dx$$

$$= \ln|x+2| - \frac{1}{2}\int \left[\frac{2x-2}{x^2-2x+4} - \frac{6}{x^2-2x+4}\right]dx$$

$$= \ln|x+2| - \frac{1}{2}\int \frac{2x-2}{(x^2+4-2x)}dx + 3\int \frac{1}{x^2-2x+4}dx$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2+4-2x| + 3\int \frac{1}{x^2-2x+1+3}dx$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\int \frac{1}{(x-1)^2+(\sqrt{3})^2}dx$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \sqrt{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$$

MULTIPLE CHOICE QUESTIONS

Q.1 $\int \frac{x^4+1}{x^2+1}dx =$

(a) $\frac{x^3}{3} - x + 2\tan x + c$ (b) $\frac{x^5}{5} - x + 2\tan^{-1}x + c$

(c) $\frac{x^3}{3} - x + 2\cot^{-1}x + c$ (d) $\frac{x^3}{3} - x + 2\tan^{-1}x + c$

EXERCISE 3.6

SHORT ANSWERS TO THE QUESTIONS

Q.1 Define Definite Integral.

(Bahawalpur Board 2019)(Faisalabad Board 2019 G-II)

Sol: If $\phi(x)$ is any anti-derivative of f(x), then the difference $\phi(b) - \phi(a)$ is called the definite integral of f(x) from a to be and is denoted by

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

Where a and b are called lower and upper limit.

Q.2 Find $\int_{-1}^{3} (x^3 + 3x^2)dx$. *(Rawalpindi Board 2019)*

(Faisalabad Board 2013, 2017, 2019 G-I)
(Gujranwala Board 2014)(Lahore Board 2018 G-II)
(Multan Board 2018 G-II)(Lahore Board 2015 G-II)

Ans. $\int_{-1}^{3} x^3\,dx + 3\int_{-1}^{3} x^2 dx$

$= \left|\frac{x^4}{4}\right|_{-1}^{3} + 3\left|\frac{x^3}{3}\right|_{-1}^{3}$

$= \frac{1}{4}((3)^4 - (-1)^4) + ((3)^3 - (-1)^3)$

$= \frac{1}{4}(81 - 1) + (27 + 1) = 20 + 28 = 48$

Q.3 Find $\int_{1}^{2} \frac{x}{x^2+2}\,dx$. *(Gujranwala Board 2017)*

(Gujranwala Board 2013) (Rawalpindi Board 2013)
(Multan Board 2018 G-I)(Sahiwal Board 2014)
(Lahore Board 2014 G-II, 2016 G-I)
(Sargodha Board 2016)

Ans. $\int_{1}^{2} \frac{x}{x^2+2}\,dx$

$= \frac{1}{2}\int_{1}^{2} \frac{2x}{x^2+2}\,dx = \frac{1}{2}\left| ln|x^2 + 2| \right|_1^2$

$= \frac{1}{2}[ln(2^2 + 2) - ln(1^2 + 2)]$

$= \frac{1}{2}[ln6 - ln3] = \frac{1}{2} ln\frac{6}{3} = \frac{1}{2} ln2$

$= ln\, 2^{\frac{1}{2}} = ln\sqrt{2}$

Q.4 Evaluate the definite integral $\int_{-1}^{1} (x^{\frac{1}{3}} + 1)dx$.

(Rawalpindi Board 2013)(Lahore Board 2019 G-II)
(Multan Board 2013, 2017 G-I)

Ans. $\int_{-1}^{1} (x^{\frac{1}{3}} + 1)dx$

$= \int_{-1}^{1} x^{\frac{1}{3}}\,dx \int_{-1}^{1} 1dx$

$= \left| \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} \right|_{-1}^{1} + |x|_{-1}^{1}$

$= \frac{3}{4}\left[((1)^4)^{\frac{1}{3}} - ((-1)^4)^{\frac{1}{3}}\right] + [1 - (-1)]$

$= \frac{3}{4}[1 - 1] + [1 + 1]$

$= \frac{3}{4}(2) = \frac{3}{2}$

Q.5 Evaluate $\int_{-1}^{1} \left(x + \frac{1}{2}\right)\sqrt{x^2 + x + 1}\,dx$.

(A.J.K Board 2017)

Ans. $\int_{-1}^{1} \left(x + \frac{1}{2}\right)\sqrt{x^2 + x + 1}\,dx$

$= \frac{1}{2}\int_{-1}^{1} (x^2 + x + 1)^{\frac{1}{2}} (2x + 1)\,dx$

$f(x) = x^2 + x + 1$

$f'(x) = (2x + 1)$

$\int [f(x)]^4 f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$

$= \frac{1}{2}\left| \frac{(x^2 + x + 1)^{\frac{3}{2}}}{\frac{3}{2}} \right|_{-1}^{1}$

$= \frac{1}{3}[(1^2 + 1 + 1)^{\frac{3}{2}} - ((-1)^2 + (-1) + 1)^{\frac{3}{2}}]$

$= \frac{1}{3}[3^{\frac{3}{2}} - 1^{\frac{3}{2}}]$

$= \frac{1}{3}[3\sqrt{3} - 1]$

Q.6 Evaluate $\int_1^2 \frac{x^2+1}{x+1}\,dx$. *(Multan Board 2013 G-I)*

Ans. $\int_1^2 \frac{x^2+1}{x+1}\,dx = \int_1^2 \frac{x^2-1+2}{x+1}\,dx$

$\int_1^2 \left(\frac{x^2-1}{x+1}+\frac{2}{x+1}\right)dx = \int_1^2 \frac{x^2-1}{x+1}\,dx + \int_1^2 \frac{2}{x+1}\,dx$

$\int_1^2 \frac{(x+1)(x-1)}{(x+1)}\,dx + 2\int_1^2 \frac{1}{x+1}\,dx$

$= \int_1^2 (x-1)dx + 2[\ln(x+1)]_1^2$

$= \left[\frac{x^2}{2}\right]_1^2 - [x]_1^2 + 2[\ln(x+1)]_1^2$

$= \left[\frac{(2)^2}{2} - \frac{(1)^2}{2}\right] - [2-1] + 2[\ln(2+1) - \ln(1+1)]$

$= \left[\frac{4}{2} - \frac{1}{2}\right] - (1) + 2[\ln 3 - \ln 2] = \left(2-\frac{1}{2}\right) - 1 + 2[\ln 3 - \ln 2]$

$\Rightarrow \frac{1}{2} + 2\ln\frac{3}{2} = \int_1^2 \frac{x^2+1}{x+1}\,dx = \frac{1}{2} + 2\ln\frac{3}{2}$ Ans

Q.7 Evaluate $\int_1^2 (x^2+1)\,dx$

(Gujranwala Board 2016)(D.G Khan Board 2017 G-I)

Sol: $\int_1^2 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_1^2$

$= \left\{\frac{(2)^3}{3}+2\right\} - \left\{\frac{(1)^3}{3}+1\right\} = \frac{8}{3}+2-\frac{1}{3}-1$

$= \frac{8-1}{3}+1 = \frac{7+3}{3} = \frac{10}{3}$

Q.8 Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos t\,dt$ (Lahore Board 2017 G-II)

(Lahore Board 2014 G-II)(Bahawalpur Board 2014) (Rawalpindi Board 2017 G-II)(Sargodha Board 2019) (Gujranwala Board 2019 G-II)(Sahiwal Board 2019)

Sol: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos t\,dt = [\sin t]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \left\{\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{6}\right)\right\}$

$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$

Q.9 Evaluate $\int_0^{\frac{\pi}{3}} \cos^2\theta \sin\theta\,d\theta$

(Rawalpindi Board 2016)(Bahawalpur Board 2016) (Lahore Board 2018 G-II)

Sol: $\int_0^{\frac{\pi}{3}} \cos^2\theta \sin\theta\,d\theta = -\int_0^{\frac{\pi}{3}} (\cos\theta)^2 \times (-\sin\theta)\,d\theta$

$= -\left[\frac{(\cos\theta)^3}{3}\right]_0^{\frac{\pi}{3}}$

$= -\frac{1}{3}\left[\cos^3\theta\right]_0^{\frac{\pi}{3}} = -\frac{1}{3}\left\{\cos^3\left(\frac{\pi}{3}\right) - \cos^3(0)\right\}$

$= -\frac{1}{3}\left\{\left(\frac{1}{2}\right)^3 - (1)^3\right\}$

$= -\frac{1}{3}\left\{\frac{1}{8} - 1\right\} = -\frac{1}{3}\left\{\frac{1-8}{8}\right\} = \frac{7}{24}$

Q.10 Evaluate $\int_0^{\pi/4} \frac{1}{1+\sin x}\,dx$

(Lahore Board 2013 G-II)(Gujranwala Board 2010)

Sol: $\int_0^{\pi/4} \frac{1}{1+\sin x}\,dx = \int_0^{\pi/4} \frac{1}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)}dx$

$= \int_0^{\pi/4} \frac{(1-\sin x)}{1-\sin^2 x}\,dx = \int_0^{\pi/4} \frac{(1-\sin x)}{\cos^2 x}dx$

$= \int_0^{\pi/4} \left[\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right]dx$

$= \int_0^{\pi/4} \sec^2 x\,dx - \int_0^{\pi/4} \sec x \tan x\,dx$

$= |\tan x|_0^{\pi/4} - |\sec x|_0^{\pi/4}$

$= \left\{\tan\frac{\pi}{4} - \tan 0\right\} - \left\{\sec\frac{\pi}{4} - \sec(0)\right\}$

$= (1-0) - (\sqrt{2}-1) = 1 - \sqrt{2} + 1 = 2 - \sqrt{2}$

Q.11 Evaluate $\int_{-6}^{2} \sqrt{3-x}\, dx$

(Bahawalpur Board 2016)(Sargodha Board 2017)

Sol: $\int_{-6}^{2} \sqrt{3-x}\, dx = -\int_{-6}^{2} \sqrt{3-x} \times (-1)\, dx$

$= -\left[\dfrac{(3-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{-6}^{2} = -\dfrac{2}{3}\left[(3-x)^{\frac{3}{2}}\right]_{-6}^{2}$

$= -\dfrac{2}{3}\left\{(3-2)^{\frac{3}{2}} - (3+6)^{\frac{3}{2}}\right\} = -\dfrac{2}{3}\left\{(1)^{\frac{3}{2}} - (9)^{\frac{3}{2}}\right\}$

$= -\dfrac{2}{3}\left\{1-(3^2)^{\frac{3}{2}}\right\}$

$= -\dfrac{2}{3}\{1-(3)^3\} = -\dfrac{2}{3}(1-27)$

$= -\dfrac{2}{3}(-26) = \dfrac{52}{3}$

Q.12 Evaluate $\int_{0}^{\frac{\pi}{6}} x \cos x\, dx$

(Lahore Board 2015 G-II)

Sol: Applying the formula

$\int x \cos x\, dx = x \sin x - \int (\sin x)\,(1)\, dx$

$= x \sin x - [(-\cos x) + c]$

$= x \sin x + \cos x + c$

Thus $\int_{0}^{\frac{\pi}{6}} x \cos x\, dx = [x \sin x + \cos x]_{0}^{\frac{\pi}{6}}$

$= \left(\dfrac{\pi}{6}\sin\dfrac{\pi}{6} + \cos\dfrac{\pi}{6}\right) - (0 \sin 0 + \cos 0)$

$= \dfrac{\pi}{6}\cdot\dfrac{1}{2} + \dfrac{\sqrt{3}}{2} - (0+1) = \dfrac{\pi}{12} + \dfrac{\sqrt{3}}{2} - 1$

Q.13 Evaluate $\int_{1}^{e} x \ln x\, dx$

(Lahore Board 2012 G-II)

Sol: Applying the formula

$\int f(x)\, \phi'(x)\, dx = f(x)\, \phi(x) - \int \phi(x)\, f'(x)\, dx$, we have

$\int (\ln x)\, x\, dx = (\ln x) \cdot \dfrac{x^2}{2} - \int \left(\dfrac{x^2}{2}\right) \cdot \dfrac{1}{x}\, dx$

$= \dfrac{1}{2}x^2 \ln x - \dfrac{1}{2}\int x\, dx = \dfrac{1}{2}x^2 \ln x - \dfrac{1}{2}\left(\dfrac{x^2}{2}\right) + c$

Thus $\int_{1}^{e} x \ln x\, dx = \left[\dfrac{1}{2}x^2 \ln x - \dfrac{x^2}{4}\right]_{1}^{e}$

$= \left(\dfrac{1}{2}e^2 \ln e - \dfrac{e^2}{4}\right) - \left(\dfrac{1}{2}(1)^2 \ln 1 - \dfrac{1^2}{4}\right)$

$= \left(\dfrac{e^2}{2}\cdot 1 - \dfrac{e^2}{4}\right) - \left(\dfrac{1}{2}(1)(0) - \dfrac{1}{4}\right)$ $(\because \ln e = 1 \text{ and } \ln 1 = 0)$

$= \dfrac{e^2}{4} + \dfrac{1}{4}$

Q.14 Evaluate $\int_{2}^{3} \left(x - \dfrac{1}{x}\right)^2 dx$

(Gujranwala Board 2011)(D.G Khan Board 2017 G-I)

Sol: $\int_{2}^{3} \left(x - \dfrac{1}{x}\right)^2 dx = \int_{2}^{3} \left(x^2 + \dfrac{1}{x^2} - 2\right) dx$

$= \left[\dfrac{x^3}{3} - \dfrac{1}{x} - 2x\right]_{2}^{3} = \left(\dfrac{27}{3} - \dfrac{1}{3} - 6\right) - \left(\dfrac{8}{3} - \dfrac{1}{2} - 4\right)$

$= \left(3 - \dfrac{1}{3}\right) - \left(\dfrac{16-3-24}{6}\right)$

$= \dfrac{9-1}{3} - \left(\dfrac{-11}{6}\right) = \dfrac{8}{3} + \dfrac{11}{6} = \dfrac{16+11}{6} = \dfrac{27}{6} = \dfrac{9}{2}$

Q.15 Evaluate $\int_{0}^{3} \dfrac{1}{x^2+9}\, dx$

(Lahore Board 2015 G-I)(Bahawalpur Board 2014)
(D.G Khan Board 2017 G-I,II)
(Multan Board 2017 G-I)(Lahore Board 2018 G-I)
(Gujranwala, Bahawalpur Board 2018)
(Faisalabad Board 2019 G-I)

Sol: $\int_{0}^{3} \dfrac{1}{x^2+9}\, dx = \int_{0}^{3} \dfrac{1}{x^2+(3)^2} dx = \dfrac{1}{3}\left[\tan^{-1}\left(\dfrac{x}{3}\right)\right]_{0}^{3}$

$= \dfrac{1}{3}\left\{\tan^{-1}\left(\dfrac{3}{3}\right) - \tan^{-1}\left(\dfrac{0}{3}\right)\right\}$

$= \dfrac{1}{3}\{\tan^{-1}(1) - \tan^{-1}(0)\} = \dfrac{1}{3}\left(\dfrac{\pi}{4} - 0\right) = \dfrac{\pi}{12}$

Q.16 Evaluate $\int_0^{\frac{\pi}{4}} \sec x (\sec x + \tan x)\, dx$.

(Lahore Board 2013)(Lahore Board 2010 G-I) (Sahiwal Board 2018)

Ans. $\int_0^{\frac{\pi}{4}} (\sec^2 x + \sec x \tan x)\, dx$

$= \int_0^{\frac{\pi}{4}} \sec^2 x (\sec x + \tan x)\, dx$

$= \int_0^{\frac{\pi}{4}} \sec^2 x\, dx + \int_0^{\frac{\pi}{4}} \sec x \tan x\, dx$

$= |\tan x|_0^{\frac{\pi}{4}} + |\sec x|_0^{\frac{\pi}{4}}$

$= \left(\tan\frac{\pi}{4} - \tan 0\right) + \left(\sec\frac{\pi}{4} - \sec 0\right)$

$= (1-0) + \sqrt{2} - 1 = 1 + \sqrt{2} - 1 = \sqrt{2}$

Q.17 Find $\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x}\, dx$.

(Faisalabad Board 2013) (Gujranwala Board 2010)

Ans. $\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x}\, dx. = \int_0^{\frac{\pi}{4}} \left(\frac{\sin x}{\cos x}\frac{1}{\cos x} - \frac{1}{\cos^2 x}\right) dx$

$\int_0^{\frac{\pi}{4}} \tan x \sec x\, dx - \int_0^{\frac{\pi}{4}} \sec^2 x dx$

$= \sec x\Big|_0^{\frac{\pi}{4}} - \tan x\Big|_0^{\frac{\pi}{4}}$

$= \left(\sec\frac{\pi}{4} - \sec 0\right) - \left(\tan\frac{\pi}{4} - \tan 0\right)$

$= (\sqrt{2} - 1) - (1 - 0) = \sqrt{2} - 2$

Q.18 $\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x}\, dx$ *(Sargodha Board 2017)*

Ans. $\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x}\, dx$

$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}\right) dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\, dx$

$= \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{\cos^2 x}\, dx = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) dx$

$= \int_0^{\frac{\pi}{4}} (\sec^2 x + \sec x \tan x)\, dx = \tan x + \sec x \int_0^{\frac{\pi}{4}}$

$= \left(\tan\frac{\pi}{4} + \sec\frac{\pi}{4}\right) - (\tan 0 + \sec 0)$

$= (1 + \sqrt{2}) - (0 + 1) = 1 + \sqrt{2} - 1 = \sqrt{2}$

Q.19 Evaluate $\int_0^{\frac{\pi}{6}} \cos^3\theta\, d\theta$

(Lahore Board 2012 G-I)(Rawalpindi Board 2014)

Sol: $\int_0^{\frac{\pi}{6}} \cos^3\theta\, d\theta = \int_0^{\frac{\pi}{6}} \cos\theta \cos^2\theta\, d\theta$

$= \int_0^{\frac{\pi}{6}} \cos\theta (1 - \sin^2\theta)\, d\theta$

$= \int_0^{\frac{\pi}{6}} \cos\theta\, d\theta - \int_0^{\frac{\pi}{6}} \cos\theta \sin^2\theta\, d\theta$

$= [\sin\theta]_0^{\frac{\pi}{6}} - \left[\frac{(\sin\theta)^3}{3}\right]_0^{\frac{\pi}{6}}$

$= \left\{\sin\left(\frac{\pi}{6}\right) - \sin(0)\right\} - \frac{1}{3}\left[\left\{\sin\left(\frac{\pi}{6}\right)\right\}^3 - \{\sin(0)\}^3\right]$

$= \left(\frac{1}{2} - 0\right) - \frac{1}{3}\left\{\left(\frac{1}{2}\right)^3 - 0\right\} = \frac{1}{2} - \frac{1}{24} = \frac{12-1}{24} = \frac{11}{24}$

Q.20 Evaluate $\int_{-1}^{2} (x + |x|)\, dx$

(Sargodha Board 2013)(Lahore Board 2015 G-I)

Sol: $\int_{-1}^{2} (x + |x|)\, dx = \int_{-1}^{0} (x + |x|)\, dx + \int_{0}^{2} (x + |x|)$

$$= \int_{-1}^{0} [x + (-x)]\,dx + \int_{0}^{2} (x + (x))\,dx$$

$$\left(\because |x| = -x \text{ if } x < 0 \right.$$
$$\left. = x \text{ if } x > 0\right.$$

$$= \int_{-1}^{0} 0\,dx + \int_{0}^{2} 2x\,dx = 0 + 2\int_{0}^{2} x\,dx$$

$$= 2\left[\frac{x2}{2}\right]_{0}^{2} = 2\left(\frac{4}{2} - \frac{0}{2}\right) = 4$$

Q.21 Evaluate $\int_{-1}^{2} (x + |x|)\,dx$.

(Gujranwala Board 2019)

Sol:
$$\int_{-1}^{2} (x + |x|)\,dx = \int_{-1}^{0} (x + |x|)\,dx + \int_{0}^{2} (x + |x|)\,dx$$

$$= \int_{-1}^{0} (x + (-x))\,dx + \int_{0}^{2} (x + x)\,dx$$

$$= \int_{-1}^{0} (x - x)\,dx + \int_{0}^{2} 2x\,dx = \int_{-1}^{0} 0\,dx + \int_{0}^{2} 2x\,dx$$

$$= 0 + \frac{2x^2}{2}\Big]_{0}^{2} = \Big[x^2\Big]_{0}^{2}$$

$$= ((2)^2 - (0)^2) = 0 + (4 - 0) = 4$$

LONG QUESTIONS

Q.1 Evaluate $\int_{-1}^{3} (x^3 + 3x^2)\,dx$. *(Gujranwala Board 2016)*

Sol: See Short Question 6

Q.2 Evaluate the definite integral $\int_{2}^{3} \left(x - \frac{1}{x}\right)^2 dx$.

(Lahore Board 2013)

Sol: See Short Question 25

Q.3 Evaluate $\int_{-1}^{1} \left(x + \frac{1}{2}\right)\sqrt{x^2 + x + 1}\,dx$.

(Gujranwala Board 2013)

Sol: See Short Question 10

Q.4 Evaluate $\int_{1}^{2} \frac{x^2 + 1}{x + 1}\,dx$. *(Multan Board 2014 G-I)*

(Lahore Board 2017 G-II)

Sol: See Short Question 11

Q.5 Evaluate $\int_{0}^{\frac{\pi}{6}} \cos^3\theta\,d\theta$. *(Sahiwal Board 2014)*

(Faisalabad Board 2017)

Sol: See Short Question 30

Q.6 Evaluate $\int_{-1}^{2} (x + |x|)\,dx$.

(Lahore Board 2014 G-II)(Sahiwal Board 2015)

Sol: See Short Question 31

Q.7 $\int_{0}^{\frac{\pi}{4}} \frac{\sin\theta + \cos\theta}{\cos 2\theta + 1}\,d\theta$ or $\int_{0}^{\frac{\pi}{4}} \frac{\sin\theta + \cos\theta}{2\cos^2\theta}\,d\theta$

(Multan Board 2015 G-II, 2016 G-I)
(Sargodha Board 2018)(Bahawalpur Board 2016)
(Rawalpindi Board 2017 G-II)(Gujranwala Board 2019 G-II)

Sol:
$$\int_{0}^{\frac{\pi}{4}} \frac{\sin\theta + \cos\theta}{1 + \cos 2\theta}\,d\theta = \int_{0}^{\frac{\pi}{4}} \frac{\cos\theta + \sin\theta}{2\cos^2\theta}\,d\theta$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{4}} \frac{\cos\theta}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta}\,d\theta = \frac{1}{2}\int_{0}^{\frac{\pi}{4}} \left(\frac{1}{\cos\theta} + \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}\right)dx$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{4}} (\sec\theta + \sec\theta\tan\theta)\,d\theta$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{4}} (\sec\theta)\,d\theta + \frac{1}{2}\int_{0}^{\frac{\pi}{4}} (\sec\theta\tan\theta)\,d\theta$$

$$= \frac{1}{2}\ |\ln|\sec\theta + \tan\theta|\Big|_{0}^{\frac{\pi}{4}} + \frac{1}{2}\ |\sec\theta|\Big|_{0}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\ln\left(\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right) - \ln(\sec 0 + \tan 0)\right] + \frac{1}{2}\left[\sec\tfrac{\pi}{4} - \sec 0\right]$$

$$= \frac{1}{2}[(\ln(\sqrt{2} + 1)) - (\ln(1 + 0))] + \frac{1}{2}[\sqrt{2} - 1]$$

$$= \frac{1}{2}[(\ln(\sqrt{2} + 1)) - (\ln 1)] + \frac{1}{2}[\sqrt{2} - 1]$$

$$= \frac{1}{2}[(\ln\sqrt{2} + 1)) - (0)] + \frac{1}{2}[\sqrt{2} - 1]$$

$$= \frac{1}{2}[\ln(\sqrt{2} + 1) + \sqrt{2} - 1]$$

Q.8 Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x}\,dx$. *(Bahawalpur Board 2014)*

Sol:
$$\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\,dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{1-\sin^2 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{\cos^2 x}\,dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x}\,dx - \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^2 x}\,dx$$
$$= \int_0^{\frac{\pi}{4}} \sec^2 x\,dx - \int_0^{\frac{\pi}{4}} \sec x \tan x\,dx$$
$$= \tan x\Big|_0^{\frac{\pi}{4}} - \sec x\Big|_0^{\frac{\pi}{4}}$$
$$= \left(\tan\frac{\pi}{4} - \tan 0\right) - \left(\sec\frac{\pi}{4} - \sec 0\right)$$
$$= (1-0) - (\sqrt{2}-1) = 1 - \sqrt{2} + 1 = 2 - \sqrt{2}$$

Q.9 Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x}\,dx$.

(Lahore Board 2015 G-I)(Faisalabad Board 2016) (Sargodha Board 2016)

Sol: See Short Question 28

Q.10 Evaluate $\int_1^e x \ln x\,dx$ *(D.G.K Board 2017 G-II)*

Sol: See Short Question 24

Q.11 Evaluate $\int_0^{\frac{\pi}{4}} \cos^4 t\,dt$ *(Lahore Board 2018 G-II)*

Sol:
$$\int_0^{\frac{\pi}{4}} \cos^4 t\,dt = \int_0^{\frac{\pi}{4}} (\cos^2 t)^2\,dt = \int_0^{\frac{\pi}{4}} \left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$\Theta\ \cos^2 t = \frac{1+\cos 2t}{2}$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} (1+\cos^2 2t + 2\cos 2t)\,dt$$
$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(1 + \left(\frac{1+\cos 4t}{2}\right) + 2\cos 2t\right) dt$$
$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(\frac{2+1+\cos 4t + 4\cos 2t}{2}\right) dt$$
$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} (3+\cos 4t + 4\cos 2t)\,dt$$
$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} (3)\,dt + \frac{1}{8}\int_0^{\frac{\pi}{4}} (\cos 4t)\,dt + \frac{1}{8}\int_0^{\frac{\pi}{4}} (4\cos 2t)\,dt$$
$$= \frac{1}{8}[3t]_0^{\frac{\pi}{4}} + \frac{1}{8}\left[\frac{\sin 4t}{4}\right]_0^{\frac{\pi}{4}} + \frac{1}{8}\left[4\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{8}\left[\left(\frac{3\pi}{4} - 0\right)\right] + \frac{1}{8}\left[\left(\frac{\sin\frac{4\pi}{4}}{4}\right) - \left(\frac{\sin 0}{4}\right)\right] + \frac{1}{8}\left[\left(2\sin\frac{2\pi}{4}\right) - \left(\frac{\sin 0}{2}\right)\right]$$
$$= \frac{1}{8}\left[\left(\frac{3\pi}{4} + \frac{\sin\pi}{4} - 0 + 2\sin\pi/2\right) - 0\right]$$
$$= \frac{1}{8}\left[\left(\frac{3\pi}{4} + 0 + 2(1)\right)\right]$$
$$= \frac{1}{8}\left[\left(\frac{3\pi}{4} + 2\right)\right] = \frac{1}{8}\left[\frac{3\pi+8}{4}\right] = \frac{3\pi+8}{32}$$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 $\int_0^{\pi/4} \sec^2 x\,dx$ equals:

(Sahiwal Board 2014) (Faisalabad Board 2016)

(a) 5 (b) 4
(c) 2 (d) 1

Q.2 $\int_{-\pi}^{\pi} \sin x\,dx$ *(D.G.K Board 2014 G-II)*

(Lahore Board 2014 G-I, II)(D.G.K Board 2015 G-II)

(a) 0 (b) 2
(c) 4 (d) 6

Q.3 If $\int_{-2}^{1} f(x)\,dx = 5$, then $\int_{1}^{-2} f(x)\,dx$ =

(Gujranwala Board 2014)

(a) –5 (b) 5
(c) 0 (d) 1

Q.4 $\int_{a}^{b} f(x)\,dx$ = ——— *(Multan Board 2014 G-II)*

(A.J.K Board 2017)

(a) $-\int_{a}^{b} f(x)\,dx$ (b) $-\int_{a}^{a} f(x)\,dx$
(c) $\int_{-b}^{-a} f(x)\,dx$ (d) $-\int_{-a}^{-b} f(x)\,dx$

Q.5 $\int_{0}^{-1} \frac{1}{1+x^2}\,dx$ is equal to: *(D.G.K Board 2011)*

(a) $\frac{\pi}{4}$ (b) $\frac{4}{\pi}$
(c) $-\frac{\pi}{4}$ (d) $-\frac{4}{\pi}$

Q.6 If $\int_{0}^{1} (4x + k)\,dx = 4$, then k will be:

(Rawalpindi Board 2013, 2014)

(a) $-\frac{1}{3}$ (b) 0
(c) 1 (d) 2

Q.7 $\int_{-1}^{3} x^3\,dx$ is equal to: *(Rawalpindi Board 2014)*

(D.G.K Board 2010)

(a) 20 (b) 80
(c) 28 (d) 18

Q.8 If $\int_{-2}^{-1} f(x)\,dx = 5$, $\int_{-1}^{1} f(x)\,dx = 3$, then $\int_{-2}^{1} f(x)\,dx$ =

(Rawalpindi Board 2014)

(a) 2 (b) 3
(c) 5 (d) 8

Q.9 $\int_{0}^{\pi/2} \cos x\,dx$ equals: *(Multan Board 2013 G-II)*

(Faisalabad Board 2017)

(a) 0 (b) –1
(c) 1 (d) –2

Q.10 $\int_{1}^{2} a^x\,dx$ equals: *(Multan Board 2013 G-I)*

(a) $\frac{a^2}{\ln a} + C$ (b) $\frac{a}{(\ln a)^2} + C$
(c) $\frac{a^2}{(\ln a)^2} + C$ (d) $\frac{a^2}{\ln a} - \frac{a}{\ln a} + C$

Q.11 $\int_{0}^{\frac{1}{\sqrt{3}}} \frac{1}{1+x^2}\,dx$ = *(Sargodha Board 2013)*

(D.G.K Board 2010)

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

Q.12 $\int_{0}^{\frac{1}{\sqrt{2}}} \frac{dx}{\sqrt{1-x^2}}$ *(Rawalpindi Board 2013)*

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) π

Q.13 If a<c<b, $\int_{a}^{b} f(x)\,dx$ = *(Faisalabad Board 2013)*

(a) $\int_{a}^{c} f(x)\,dx$ (b) $\int_{c}^{b} f(x)\,dx$
(c) $\int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$
(d) $\int_{a}^{c} f(x)\,dx - \int_{c}^{b} f(x)\,dx$

Q.14 $\int_{-\pi}^{\pi} \cos x\,dx = 0$. *(D.G.K Board 2015 G-I)*

(a) 0 (b) 2
(c) –2 (d) 1

Q.15 $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ = *(D.G.K Board 2015 G-I)*

(Multan Board 2017 G-I)(Rawalpindi Board 2016)

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) 1 (d) None of these

Q.16 $\int_{0}^{1} \frac{1}{1+x^2}\,dx$ = ? *(D.G.K Board 2013 G-I)*

(Lahore Board 2017 G-I)(Gujranwala Board 2018)

(a) 0 (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

Q.17 $\int_{0}^{4} x\,dx$ = ? *(D.G.K Board 2013 G-I)*

(a) 0 (b) 6
(c) 8 (d) 16

Q.18 $\int_0^{\pi/2} \cos 2x\, dx$ equals to: *(D.G.K Board 2010)*

(a) 1 (b) 2
(c) −2 (d) 0

Q.19 If $\int_a^b f(x)\, dx$, then 'b' is known as the ---------- of integration.
(a) Domain (b) Range
(c) Lower limit (d) Upper limit

Q.20 If $\int_a^b f(x)\, dx$, then 'a' is known as the -------- of integration.
(a) Domain (b) Range
(c) Lower limit (d) Upper limit

Q.21 If $\int f(x)\, dx = \phi(x) + c$, as c is not definite, so $\phi(x) + c$ is called the ---------- of f(x).
(a) Integral (b) Indefinite integral
(c) Differential (d) None of these

Q.22 $\int_0^{\sqrt{3}} \frac{1}{1+x^2}\, dx =$

(a) $\frac{\pi}{6}$ (b) π
(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

Q.23 If $\int_{-2}^{1} f(x)\, dx = 5$, $\int_1^3 f(x)dx = 3$, then $\int_{-2}^{3} f(x)dx = ?$ *(Lahore Board 2011 G-I)*

(a) 8 (b) 5
(c) 3 (d) 1

Q.24 $\int_0^3 \frac{1}{x^2+9}\, dx$ equals. *(Lahore Board 2014 G-I)*

(a) $\frac{12}{\pi}$ (b) $\frac{\pi}{12}$
(c) $-\frac{12}{\pi}$ (d) $-\frac{\pi}{12}$

Q.25 $\int_0^{\pi/4} \text{Sec } x \tan x\, dx =$ --------

(Gujranwala Board 2013)

(a) $\sqrt{2}$ (b) $\sqrt{2} - 1$
(c) $\sqrt{2} + 1$ (d) 1

Q.26 $\int_{-1}^{1} \frac{1}{x\sqrt{x^2-1}}\, dx =$ *(Lahore Board 2013 G-II)*

(a) $-\pi$ (b) $\frac{\pi}{3}$
(c) π (d) $\frac{\pi}{4}$

Q.27 $\int_0^1 \frac{dx}{\sqrt{1-x^2}} =$ *(Lahore Board 2013 G-I)*

(Rawalpindi Board 2018)

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

Q.28 $\int_{-\pi/4}^{\pi/4} \text{Sec}^2 x\, dx =$ *(Gujranwala Board 2012)*

(a) 0 (b) $\sqrt{2}$
(c) 2 (d) $\frac{1}{2\sqrt{2}}$

Q.29 $\int_0^{\pi/2} \text{Sin } x\, dx =$ *(Lahore Board 2012 G-II)*

(a) 0 (b) 1
(c) $\frac{\pi}{2}$ (d) $-\frac{\pi}{2}$

Q.30 $\int_0^{\pi} \text{Sin } x\, dx =$ *(Lahore Board 2012 G-I)*

(a) 0 (b) 2
(c) −2 (d) 1

Q.31 $\int_1^4 3\sqrt{x}\, dx$ is equal to: *(Lahore Board 2015 G-II)*

(a) 1 (b) 4
(c) 14 (d) 41

Q.32 $\int_0^1 x^3\, dx$ is equal to: *(Lahore Board 2015 G-I)*

(a) 4 (b) −4
(c) $\frac{1}{4}$ (d) $-\frac{1}{4}$

Q.33 $\int_0^1 \frac{1}{1+x^2}\, dx =$ *(Bahawalpur Board 2016)*

(a) $\frac{\pi}{4}$ (b) $\frac{2\pi}{3}$
(c) $\frac{3\pi}{4}$ (d) π

Q.34 $\int_0^{\pi/4} \text{Sec}^2 x\, dx =$ *(Faisalabad Board 2016)*

(a) 1 (b) $\frac{1}{\sqrt{2}}$
(c) $\sqrt{2}$ (d) 0

Q.35 $\int_0^{\pi/4} \frac{\text{Sec}^2 x\, dx}{1+\tan x}$ *(Lahore Board 2016 G-I)*

(Bahawalpur Board 2018)

(a) 1 (b) 2
(c) ln2 (d) $\ln\sqrt{2}$

Q.36 $\int_0^{\pi/2} k\text{Cos}x\, dx = 4$, then k = *(Rawalpindi Board 2016)*

(a) 5 (b) 4
(c) 2 (d) 0

Q.37 $\int_0^{\pi} \text{Cos}x\, dx =$ *(Sargodha Board 2016)*

(Multan Board 2018 G-II)

(a) −2 (b) −1
(c) 0 (d) 2

Q.38 $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx =$ ----------; where a < b < c.

(a) $\int_a^b f(x)\,dx$ (b) $\int_b^c f(x)\,dx$
(c) $\int_a^c f(x)\,dx$ (d) $\int_c^a f(x)\,dx$

Q.39 If $\int_1^2 f(x)\,dx = 3$, $\int_2^5 f(x)\,dx = 7$, then $\int_1^5 f(x)\,dx =$

(a) 7 (b) 10
(c) 3 (d) −10

Q.40 If $\int_{-2}^{1} f(x)\,dx = 5$, then $\int_1^{-2} f(x)\,dx =$

(Gujranwala Board 2014)

(a) −5 (b) 5
(c) 0 (d) 1

Q.41 $\int_a^x (3t^2)\,dt =$ *(D.G.K Board 2017 G-I)*

(a) $x^3 + a^3$ (b) $x^3 - a^3$
(c) $a^3 - x^3$ (d) $x^2 + a^2$

Q.42 $\int_0^{\frac{\pi}{2}} \sin^3 x \cos x\, dx =$ *(Lahore Board 2017 G-II)*

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$
(c) $\frac{1}{4}$ (d) $\frac{1}{9}$

Q.43 $\int_0^1 (3-x)\,dx$ *(Rawalpindi Board 2017 G-I)*

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$
(c) $\frac{5}{2}$ (d) $\frac{2}{5}$

Q.44 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx =$ *(Rawalpindi Board 2017 G-II)*

(a) 0 (b) 1
(c) 2 (d) −2

Q.45 $\int_a^b x\,dx$ equal: *(Lahore Board 2015 G-II)*

(a) $\frac{b-a}{2}$ (b) $\frac{b+a}{2}$
(c) $\frac{b^2-a^2}{2}$ (d) $\frac{b^2+a^2}{2}$

Q.46 $\int_0^2 (x^2+1)\,dx =$ *(A.J.K Board 2017)*

(Rawalpindi Board 2018)

(a) $\frac{3}{10}$ (b) 2
(c) $\frac{14}{3}$ (d) 0

Q.47 $\int_{-1}^{2} x\,dx =$ *(Faisalabad Board 2018)*

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$
(c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

Q.48 $\int_0^{\frac{\pi}{2}} \cos x\, dx =$ *(Lahore Board 2018 G-I)*

(a) 0 (b) 1
(c) 2 (d) 3

Q.49 $3\int_{\frac{\pi}{2}}^{\pi} \sin x\, dx =$ *(Multan Board 2018 G-I)*

(a) 1 (b) 2
(c) 3 (d) 4

Q.50 $\int_2^4 \frac{1}{x}\,dx =$ *(Multan Board 2018 G-II)*

(a) ln4 (b) 4
(c) ln2 (d) 2

Q.51 If $\int_2^k 2\,dx = 12$, then K = ?

(Bahawalpur Board 2019)

(a) 12 (b) 16
(c) 8 (d) 4

Q.52 $\int_0^x 3t^2\,dt =$: *(Faisalabad Board 2019)*

(a) t^3 (b) $\frac{t^3}{3}$
(c) x^3 (d) 0

Q.53 $\int_0^1 \frac{1}{1+x^2}\,dx =$: *(Faisalabad Board 2019 G-II)*

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

Q.54 $\int_a^b 3t^2\,dat =$ *(Gujranwala Board 2019 G-I)*

(a) $a^3 - b^3$ (b) $a^3 + b^3$
(c) $b^3 - a^3$ (d) $\frac{b^3 + a^3}{3}$

Q.55 $\int_{-1}^0 \frac{1}{1+x^2}\,dx =$ *(Gujranwala Board 2019 G-I)*

(a) $\frac{\pi}{4}$ (b) $\frac{4}{\pi}$
(c) $-\frac{\pi}{4}$ (d) $-\frac{4}{\pi}$

Q.56 $\int_0^\pi \sec x \tan x\,dx =$

(Gujranwala Board 2019 G-II)

(a) 0 (b) 1
(c) −1 (d) −2

Q.57 $\int_{-\pi}^{\pi} \sin x\,dx =$: *(Lahore Board 2019)*

(a) 2π (b) 0
(c) 1 (d) $\cos\pi$

Q.58 $\int_1^2 2x\,dx =$ *(Multan Board 2019)*

(a) 3 (b) 2
(c) 1 (d) 0

Q.59 $\int_1^2 \frac{1}{x}\,dx =$ *(Multan Board 2019)*

(a) $2\lambda nx$ (b) $\lambda n2$
(c) λn (d) $\lambda n3$

Q.60 $\int_0^{\pi/2} \text{Cos t}\,dt$ will be equal to

(Sargodha Board 2019)

(a) 1 (b) 2
(c) 0 (d) 3

Q.61 Area bounded by curve y = f(x) about X - axis from x = a to x = b is denoted by

(Sargodha Board 2019)

(a) $\int_a^b x\,dy$ (b) $\int_a^b x\,dx$
(c) $\int_a^b y\,dx$ (d) $\int_a^b y\,dx$

Q.62 If $\int_{-1}^5 f(x)\,dx = 5$, then $\int_5^{-1} f(x)\,dx =$

(Sahiwal Board 2019)

(a) $\frac{1}{5}$ (b) $-\frac{1}{5}$
(c) −5 (d) 5

Q.63 $\int \lambda nx\,dx =$ *(Rawalpindi Board 2019)*

(a) −1 (b) 0
(c) 1 (d) e

EXERCISE 3.7

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find the area between the x-axis and the curve $y = x^2 + 1$ from x = 1 to x = 2.

(Gujranwala Board 2014)(Bahawalpur Board 2014) (Sargodha Board 2016)(Sahiwal Board 2019)

Sol: The required area $= \int_1^2 (x^2 + 1)\,dx$

$= \int_1^2 x^2\,dx + \int_1^2 1\,dx$

$= \left|\frac{x^3}{3}\right|_1^2 + |x|_1^2 = \left(\frac{2^3}{3} - \frac{1^3}{3}\right) + (2 - 1)$

$= \left(\frac{8}{3} - \frac{1}{3}\right) + 1 = \frac{7}{3} + 1 = \frac{10}{3}$ square units

Q.2 Find the area between x-axis and the curve y = x^3 + 1 from x = 1 to x = 2.

(Faisalabad Board 2013)(Lahore Board 2015 G-II)

Ans. Required Area = $\int_1^2 (x^3 + 1)dx$

$= \int_1^2 x^3\,dx + \int_1^2 dx = \left|\frac{x^4}{4}\right|_1^2 + |x|_1^2$

$= \frac{1}{4}[2^4 - 1^4] + [2 - 1] = \frac{1}{4}[16 - 1] + 1$

$= \frac{15}{4} + 1 = \frac{19}{4}$

Q.3 Find the area below the curve y = $3\sqrt{x}$ and above the x-axis between x = 1 and x = 4.

(Multan Board 2014 G-II)

Sol: The required area = $\int_1^4 3\sqrt{x}\,dx = 3\int_1^4 x^{\frac{1}{2}}\,dx$

$= 3\left|\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^4 = 3\left|\frac{2}{3}x^{\frac{3}{2}}\right|_1^4$

$= 2\left|x^{\frac{3}{2}}\right|_1^4 = 2[43/2 - 13/2]$

$= 2[(22)3/2 - 1] = 2[8 - 1] = 2(7) = 14$ square units

Q.4 Find the area above the x-axis and under the curve y = 5 – x^2 from x = –1 to x = 2.

(Lahore Board 2016 G-I, 2019 G-II)
(Sahiwal Board 2014 G-II)(Multan Board 2017 G-I)
(Rawalpindi Board 2016)(Bahawalpur Board 2019)

Sol: The required area = $\int_{-1}^{2} (5 - x^2)\,dx$

$= |5x|_{-1}^{2} - \left|\frac{x^3}{3}\right|_{-1}^{2}$

$= [5(2) - 5(-1)] - \left[\frac{2^3}{3} - \frac{(-1)^3}{3}\right]$

$= 10 + 5 - \frac{8+1}{3} = 15 - \frac{9}{3}$

$= 15 - 3 = 12$ square units

Q.5 Find the area bounded by the curve y = $4 - x^2$ and x-axis.

(Lahore Board 2012 G-I) (Multan Board 2018 G-I)

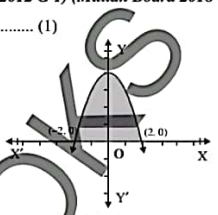
Sol: $y = 4 - x^2$ ..... (1)

First we find the points where the curve cuts the x-axis. Putting y = 0 in (1), we have

$0 = 4 - x^2$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$



So the curve cuts the x-axis at points (2, 0) and (–2, 0).

The area above the x-axis and under the curve $y = 4 - x^2$ is shown in the figure as shaded region.

Thus the required area = $\int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} (4 - x^2)\,dx$

$= \left|4x - \frac{x^3}{3}\right|_{-2}^{2} = \left[8 - \frac{(2)^3}{3}\right] - \left[4(-2) - \frac{(-2)^3}{3}\right]$

$= \left[8 - \frac{8}{3}\right] - \left[-8 + \frac{8}{3}\right]$

$= 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{8}{3} - \frac{8}{3}$

$= \frac{48 - 8 - 8}{3} = \frac{32}{3}$ square units.

Q.6 Find area between x-axis and curve y = 4x – x^2.

(Sargodha Board 2013)(Lahore Board 2017, 2018 G-II)
(Gujranwala Board 2016, 2019)

Ans. For limit Put y = 0 or 4x – x^2 = 0 or x = 0, 4

So, Required Area = $\int_0^4 y\,dx$

$= \int_0^4 4x\,dx - \int_0^4 x^2 dx = 4\left|\frac{x^2}{2}\right|_0^4 - \left|\frac{x^3}{3}\right|_0^4$

$= 2((4)^2 - 0) - \frac{1}{3}((4)^3 - 0) = 2(16) - \frac{64}{3}$

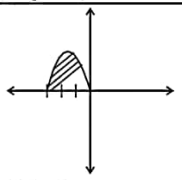
$= \frac{96 - 64}{3} = \frac{32}{3}$ sq unit

Q.7 Find the area under $y = x^3 + 3x^2$ above the x-axis between x = 0 and x = –3.

(A.J.K Board 2017)(Sahiwal Board 2018)

Ans.

x	0	–1	–2	–3
y	0	2	4	0



As $y \geq 0$ for $-3 \leq x \leq 0$

Area $= \int_{-3}^{0} y\,dx$

$= \int_{-3}^{0} (x^3 + 3x^2)\,dx$

$= \frac{x^4}{4} + x^3 \Big|_{-3}^{0} = \{0 + 0\} - \left\{\frac{81}{4} - 27\right\}$

$= -\frac{81}{4} + 27 = \frac{-81 + 108}{4} = \frac{27}{4}$

LONG QUESTIONS

Q.1 Find the area between the x-axis and the curve $y = x^2 + 1$ from $x = 1$ to $x = 2$

(Lahore Board 2013 G-II)

Sol: See Short Question 1

Q.2 Find the area bounded by the curve $y = x^3 - 4x$ and x-axis. *(Faisalabad Board 2019 G-I)*

(Lahore Board 2012 G-II, 2018 G-I)

(Multan Board 2018 G-II)

Sol: $y = x^3 - 4x = x(x^2 - 4)$

Put $y = 0$ we have $(x)(x^2 - 4) = 0$

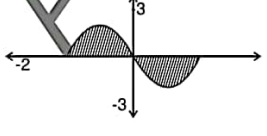
$\Rightarrow x = 0, x^2 - 4 = 0 \Rightarrow (x + 2)(x - 2) = 0$

$\Rightarrow x = -2$ and $x = 2$

x	−2	−1	0	1	2
y	0	3	0	−3	0

As $y \geq 0$ for $-2 \leq x \leq 0$ therefore the curve is above the x-axis as $y \leq 0$ for $0 \leq x \leq 2$ therefore the curve is below the x-axis.

Area $= \int_{-2}^{0} (x^3 - 4x)dx - \int_{0}^{2} (x^3 - 4x)dx$



$= \left|\frac{x^4}{4} - 4\frac{x^2}{2}\right|_{-2}^{0} - \left|\frac{x^4}{4} - 4\frac{x^2}{2}\right|_{0}^{2}$

$= \left|\frac{x^4}{4} - 2x^2\right|_{-2}^{0} - \left|\frac{x^4}{4} - 2x^2\right|_{0}^{2}$

$= \left[\left(\frac{(0)^4}{4} - 2(0)^2\right) - \left(\frac{(-2)^4}{4} - 2(-2)^2\right)\right]$

$- \left[\left(\frac{(2)^4}{4} - 2(2)^2\right) - \left(\frac{(0)^4}{4} - 2(0)^2\right)\right]$

$= \left(0 - \left(\frac{16}{4} - 2(4)\right)\right) - \left(\left(\frac{16}{4} - 2(4)\right) - 0\right)$

$= -(4 - 8) - (4 - 8) = 4 + 4 = 8$ sq. unit

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 The area of curve $y = \cos x$ from $x = -\frac{\pi}{2}$ to $\frac{\pi}{2}$ is:

(Sargodha Board 2013)

(a) 1 (b) 2

(c) 3 (d) 4

Q.2 The area of the region, above the x-axis and under the curve $y = f(x)$ from a to b is given by ---------.

(a) $\int_{a}^{b} f(x)\,dx$ (b) $-\int_{a}^{b} f(x)\,dx$

(c) $\int_{b}^{a} f(x)\,dx$ (d) None of these

Q.3 $\int_{a}^{b} f(x)\,dx$ as the area under the curve $y = f(x)$ from $x = a$ to $x = b$. and the x-axis is called.

(a) Integration by parts (b) Definite integral

(c) Differentiation (d) None of these

Q.4 The area bounded by cosx function from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$ is. *(Gujranwala Board 2013)*

(a) 1 sq unit (b) 2 sq unit

(c) 3 sq unit (d) 4 sq unit

Q.5 The area between the x-axis and the curve $y = \text{Cos}\,\frac{1}{2}x$ from $x = -\pi$ to $x = \pi$ is.

(Lahore Board 2013 G-II)

(a) 1 (b) 2

(c) 3 (d) 4

Q.6 If the graph of f is entirely below the x-axis, then the definite integral is:

(a) Positive (b) Positive or negative

(c) Negative (d) Positive and negative

EXERCISE 3.8

SHORT ANSWERS TO THE QUESTIONS

Q.1 Show that $y = cx - 1$, is the solution of the differential equation

$$x\frac{dy}{dx} = 1 + y$$ *(Sargodha Board 2016)*

Sol: $y = cx - 1$

$\Rightarrow \quad y + 1 = cx$

Differentiating both sides w.r.t. 'x' we have

$$\frac{dy}{dx} = c$$

Multiplying both sides by x

$$x\frac{dy}{dx} = x\,c$$

$$x\frac{dy}{dx} = x\left(\frac{y+1}{x}\right)\left[\because y = cx - 1 \Rightarrow c = \frac{y+1}{x}\right]$$

$$x\frac{dy}{dx} = y + 1$$

Q.2 Show that $y = c\,e^{x^2}$, is the solution of the differential equation

$$\frac{1}{x}\frac{dy}{dx} - 2y = 0$$ *(Lahore Board 2013 G-I)*

Sol: $y = c\,e^{x^2}$

Differentiating both sides w.r.t. 'x' we have

$$\frac{dy}{dx} = c\,e^{x^2} \times 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = 2\,c\,x\,e^{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2\left(\frac{y}{e^{x^2}}\right)x\,e^{x^2} \quad \left[\because y = c\,ex2 \Rightarrow c = \frac{y}{ex2}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = 2xy \Rightarrow \frac{1}{x}\cdot\frac{dy}{dx} = 2y \Rightarrow \frac{1}{x}\cdot\frac{dy}{dx} - 2y = 0$$

Q.3 Solve the differential equation: $\frac{dy}{dx} = -y$

(Lahore Board 2014 G-I)(Bahawalpur Board 2014) (D.G Khan Board 2014 G-I)(Sahiwal Board 2019)

Sol: $\frac{dy}{dx} = -y$ Separating variables, we have

$$\frac{1}{y}dy = -dx$$ Integrating both sides, we have

$$\int\frac{1}{y}dy = -\int 1\,dx$$

$$\Rightarrow \quad \ln y = -x + c_1 \Rightarrow y = e^{-x+c1}$$

$$\Rightarrow \quad y = e^{-x}.e^{c1} = c\,e^{-x}$$

Q.4 Solve the differential equation $\frac{dy}{dx} = \frac{1-x}{y}$

(Faisalabad Board 2013) (Gujranwala Board 2014) (Bahawalpur Board 2016)(Rawalpindi Board 2017 G-II)

Ans. $\frac{dy}{dx} = \frac{1-x}{y}$

$$y\,dy = (1-x)\,dx$$

By integrating

$$\int y dy = \int (1 - dx)dx$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + \frac{c}{2}$$

$$y^2 = 2x - x^2 + c$$

Q.5 Solve the differential equation $\frac{1}{x}\frac{dy}{dx} - 2y = 0.x \neq 0.y > 0$.

(Lahore Board 2013)(A.J.K Board 2017) (Faisalabad Board 2017)(D.G Khan Board 2017 G-I)

Ans. $\frac{1}{x}\frac{dy}{dx} - 2y = 0$

Separating the variables, we get.

$$\frac{1}{x}\frac{dy}{dx} = 2y$$

$$\frac{1}{y}dy = 2x\,dx$$

Integrating, we have.

$$\ln y = 2\frac{x^2}{2} + c$$

$$\ln y = x^2 + c$$

Q.6 Solve the differential equation:

$y\,dx + x\,dy = 0$

(Multan Board 2014 G-I)(Sargodha Board 2016) (Sahiwal Board 2018)(Bahawalpur Board 2019)

Sol: $y\,dx + x\,dy = 0$

$$y\,dx = -x\,dy$$

Separating variables, we have

$$\frac{1}{x}dx = -\frac{1}{y}dy$$

Integrating both sides, we have

$$\int\frac{1}{x}dx = -\int\frac{1}{y}dy$$

$$\ln x = -\ln y + \ln c \Rightarrow \ln x + \ln y = \ln c$$

$$\ln xy = \ln c \Rightarrow xy = c$$

Q.7 Solve the differential equation:

$$\frac{dy}{dx} = \frac{y}{x^2}, \quad (y > 0)$$

(Lahore Board 2014 G-II)(Multan Board 2014 G-II)
(Sahiwal Board 2013)

Sol: $\frac{dy}{dx} = \frac{y}{x^2}$

Separating variables we have

$$\frac{1}{y}dy = \frac{1}{x^2}dx$$

Integrating both sides, we have

$$\int\frac{1}{y}dy = \int\frac{1}{x^2}dx \Rightarrow \ln y = \int x^{-2}(1)\,dx$$

$$\ln y = \frac{x^{-2+1}}{-2+1} + c1 \Rightarrow \ln y = -\frac{1}{x} + c_1$$

$$y = e^{-\frac{1}{x}+c1} = e^{-\frac{1}{x}}.e^{c1} \Rightarrow y = c\,e^{-\frac{1}{x}}$$

Q.8 Solve the differential equation

$$x^2(2y+1)\frac{dy}{dx} - 1 = 0$$

(Lahore Board 2015 G-I)

Sol: The given differential equation can be written as

$$x^2(2y+1)\frac{dy}{dx} = 1 \quad \ldots\ldots (i)$$

Dividing by x^2, we have

$$(2y+1)\frac{dy}{dx} = \frac{1}{x^2}\ (x \neq 0) \quad \ldots\ldots (ii)$$

Multiplying both sides of (i) by dx, we get

$$(2y+1)\left(\frac{dy}{dx}dx\right) = \frac{1}{x^2}dx$$

or $(2y+1)\,dy = \frac{1}{x^2}dx \quad \left(\Theta \frac{dy}{dx}dx = dy\right)$

Integrating either side gives

$$\int(2y+1)\,dy = \int\frac{1}{x^2}dx$$

or $y^2 + y = -\frac{1}{x} + c \quad \left(\Theta \int x^{-2}dx = \frac{x^{-1}}{-1} + c\right)$

Thus $y^2 + y = c - \frac{1}{x}$ is the general solution of the given differential equation.

Q.9 Solve the differential equation:

$$(e^x + e^{-x})\frac{dy}{dx} = (e^x - e^{-x})$$

(Lahore Board 2012 G-I)(Rawalpindi Board 2014)

Sol: $(e^x + e^{-x})\frac{dy}{dx} = (e^x - e^{-x})$

Separating variables, we have

$$dy = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)dx$$

Integrating both sides, we have

$$\int dy = \int\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)dx$$

$$y = \ln(e^x + e^{-x}) + c$$

$$\left[\Theta \frac{d}{dx}(e^x + e^{-x}) = e^x - e^{-x}\right]$$

Q.10 Solve the differential equation.

$$\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}$$ ***(Lahore Board 2017 G-II)***

Ans. $\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}$

Separating the variable

$$\frac{x^2+1}{x}dx = \frac{y+1}{y}dy$$

Integrating

$$\int\left(x + \frac{1}{x}\right)dx = \int\left(1 + \frac{1}{y}\right)dy$$

$$\frac{x^2}{2} + \ln|x| = y + \ln|y| + C$$

Q.11 Solve the differential equation:

$$(e^x + e^{-x})\frac{dy}{dx} = (e^x - e^{-x})$$

(Lahore Board 2012 G-I)(Rawalpindi Board 2014)

Sol: $(e^x + e^{-x})\frac{dy}{dx} = (e^x - e^{-x})$

Separating variables, we have

$$dy = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)dx$$

Integrating both sides, we have

$$\int dy = \int\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)dx$$

$$y = \ln(e^x + e^{-x}) + c$$

$$\left[\Theta \frac{d}{dx}(e^x + e^{-x}) = e^x - e^{-x}\right]$$

Q.12 Solve the differential equation

$$\sec x + \tan y \frac{dy}{dx} = 0$$ *(Sahiwal Board 2018)*

(D.G Khan Board 2017 G-II)

Ans.

$$\sec x + \tan y \frac{dy}{dx} = 0$$

Separating the variable, we get

$$\tan y \frac{dy}{dx} = -\sec x$$

$$-\tan y\, dy = \sec x\, dx$$

$$-\int \tan y\, dy = \int \sec x\, dx$$

$$\int \frac{-\sin y}{\cos x} dy = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$$

$$\ln|\cos x\, y| = \ln|\sec x + \tan x| + \ln C$$

$$\ln|\cos y| = \ln|C(\sec x + \tan x)|$$

$$\cos y = C(\sec x + \tan x)$$

Q.13 Solve $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

(Multan Board 2018 G-I)

Sol: $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

$$\sec^2 x \tan y\, dx = -\sec^2 y \tan x\, dy$$

$$\frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y\, dy}{\tan y}$$

$$\frac{\sec^2 y\, dy}{\tan y} = -\frac{\sec^2 x\, dx}{\tan x}$$

Integrating

$$\int \frac{\sec^2 y\, dy}{\tan y} = -\int \frac{\sec^2 x\, dx}{\tan x}$$

$$\ln|\tan y| = -\ln|\tan x| + \ln C$$

$$\ln|\tan y| + \ln|\tan x| = \ln C$$

$$\ln|\tan y \tan x| = \ln C$$

$$\tan y \tan x = C$$

LONG QUESTIONS

Q.1 Solve the differential equation $\frac{x^2+1}{y+1} = \frac{x}{y} \cdot \frac{dy}{dx}$ where $x > 0$, $y > 0$. *(Faisalabad Board 2013)*

Sol:

$$\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}$$

$$\frac{(x^2+1)\,dx}{x} = \frac{y+1}{y} dy$$

$$\left(x + \frac{1}{x}\right) dx = \left(1 + \frac{1}{y}\right) dy$$

Integrating

$$\int \left(x + \frac{1}{x}\right) dx = \int \left(1 + \frac{1}{y}\right) dy$$

$$\frac{x^2}{2} + \ln|x| = y + \ln|y| + C$$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 The order of differential equation

$$x\frac{d^2y}{dy^2} + \frac{dy}{dx} - 2 = 0 \text{ is:}$$ *(D.G.K Board 2014 G-II)*

(a) 1 (b) 2

(c) 3 (d) 4

Q.2 The solution of differential equation $\frac{dy}{dx} = -y$ is:

(Faisalabad Board 2013, Multan Board 2014 G-II)
(Sargodha Board 2016)(Lahore Board 2017 G-II)
(Multan Board 2017 G-I)(Bahawalpur Board 2018)

(a) $y = x e^{-x}$ (b) $y = ce^{-1}$

(c) $y = e^{x}$ (d) $y = ce^{x}$

Q.3 The solution of the differential equation $y\,dx + x\,dy = 0$ is: *(Lahore Board 2014 G-I)*
(Rawalpindi Board 2013, Bahawalpur Board 2014)
(Rawalpindi Board 2017 G-II)(Gujranwala Board 2012)

(a) $\ln xy = 0$ (b) $\ln\frac{x}{y} = C$

(c) $xy = C$ (d) $\ln\frac{y}{x} = C$

Q.4 The degree of differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 3x = 0 \text{ is:}$$ *(Rawalpindi Board 2014)*

(a) 1 (b) 2

(c) 0 (d) 3

Q.5 Solution of differential equation $\frac{dy}{dx} = \cos x$ is:

(Multan Board 2013 G-I)(Rawalpindi Board 2018)

(a) $y = \cos x + C$ (b) $y = \tan x + C$

(c) $y = \sin x + C$ (d) $y = \cot x + C$

Q.6 The general solution of differential equation:

$$\sqrt{1-x^2}\,dy + dx = 0$$

(Sargodha Board 2013)

(a) $y = \tan^{-1} x + C$ (b) $y = \ln\left(x + \sqrt{1+x^2}\right) + C$

(c) $y = \cos^{-1} x + C$ (d) $y = \cot^{-1} x + C$

Q.7 $y = Ce^{x^2}$ is solution of:

(a) $\frac{1}{x}\frac{dy}{dx} - 2y = 0$ (b) $x\frac{dy}{dx} - 2y = 0$

(c) $\frac{dy}{dx} - 2y = 0$ (d) $\frac{1}{x}\frac{dy}{dx} - y = 0$

Q.8 $\sin y \operatorname{cosec} x \frac{dy}{dx} = 1$ *(D.G.K Board 2015 G-II)*

(a) $\cos y = \cos x + C$ (b) $y = x + C$

(c) $\cos x = \cos x + C$ (d) $-\cos y = -\cos y + C$

Q.9 Solution of differential equation $\frac{dy}{dx} = y$ is:

(D.G.K Board 2015 G-I)(Lahore Board 2015 G-I)

(a) $y = C e^{-x}$ (b) $y = c e^{x}$

(c) $y = e^{cx}$ (d) $y = xe^{-x}$

Q.10 Order of the differential equation $x \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} - 2y = 0$ is: *(D.G.K Board 2013 G-II)*

(a) 1 (b) 2

(c) 3 (d) 4

Q.11 The solution of differential equation $\frac{dy}{dx} = \cos x$ is: *(D.G.K Board 2010)*

(a) $y = \cos x + C$ (b) $y = \tan x + C$

(c) $y = \cot x + C$ (d) $y = \sin x + C$

Q.12 If $f'(x) = 2x - 3$ and $f(0) = -3$, then $f(2) =$

(a) -5 (b) 0

(c) -3 (d) 2

Q.13 $\int e^{1 + \ln x} dx =$ ---------

(a) $e^{1 + \ln x}$ (b) $\frac{1}{x} e^{1 + \ln x}$

(c) $\frac{e}{2} x^2$ (d) $e x^2$

Q.14 The general solution of differential equation of order n constains n arbitrary constants, which can be determined by ---------- initial value conditions.

(a) 0 (b) 1

(c) 2 (d) n

Q.15 The term dy (or df) = f'(x) δx is called the ------- of the dependent variable y.

(a) Differentiation (b) Integration

(c) Differential (d) None of these

Q.16 The area of the region, below the x-axis and under the curve y = f(x) from a to b is given by:

(a) $\int_a^b f(x)\,dx$ (b) $-\int_a^b f(x)\,dx$

(c) $-\int_b^a f(x)\,dx$ (d) None of these

Q.17 The order of a differential equation $x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 2x = 0$ is: *(D.G.K Board 2017 G-I)*

(a) 0 (b) 1

(c) 2 (d) None of these

Q.18 The arbitrary constants involving in the solution of differential equations can be determined by the given condition. Such conditions are called ------ condition.

(a) Initial values (b) General

(c) Boundary values (d) None of these

Q.19 The differential co-efficient of $e^{\sin x}$ equals.

(Lahore Board 2014 G-I)

(a) $e^{\sin x} \operatorname{Cos} x$ (b) $e^{\sin x} \sin x$

(c) $\operatorname{Sin} x\, e^{\sin x - 1}$ (d) $\operatorname{Sin} x\, e^{\sin x + 1}$

Q.20 The solution of differential equation $\frac{dy}{dx} = \operatorname{cosec} x \cot x$ is: *(Gujranwala Board 2013)*

(Gujranwala Board 2016)

(a) $y = \operatorname{Cosec} x + c$ (b) $y = \operatorname{Sec} x + c$

(c) $y = -\operatorname{Cosec} x + c$ (d) $y = -\operatorname{Cot} x + c$

Q.21 $\int \frac{2x\,dx}{\sqrt{1 - x^2}} =$ --------- *(Gujranwala Board 2013)*

(a) $\ln|1 - x^2| + c$ (b) $-2\sqrt{1 - x^2} + c$

(c) $2\sqrt{1 - x^2} + c$ (d) $\sqrt{1 - x^2} + c$

Q.22 Degree of differential equation $x^2 \left(\frac{dy}{dx}\right)^3 + 5\frac{d^2y}{dx^2} + 7 \left(\frac{d^3y}{dx^3}\right)^2 = 0$ is *(Lahore board 2012 G-I)*

(a) 3 (b) 1

(c) 2 (d) 5

Q.23 Applying initial value conditions in solution of differential equation, we get.

(Sargodha Board 2017)

(a) General solution (b) Particular solution

(c) No. solution (d) Infinite solution

Q.24 The solution of differential equation $\frac{dy}{dx} = \operatorname{Sec}^2 x$ is: *(Lahore 2016 G-I)*

(a) $y = \operatorname{Cos} x + c$ (b) $y = \operatorname{Sec} x + c$

(c) $y = \operatorname{Cos}^2 x + c$ (d) $y = \tan x + c$

Q.25 The order of a differential equation

$y\frac{dy}{dx} + 2x = 0$ is:

(a) 0 (b) 1

(c) 2 (d) None of these

Q.26 The order of the differential equation.

$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2x = 0$

(Lahore Board 2018 G-I)

(a) 2 (b) 1

(c) 0 (d) 3

Q.27 Solution of differential equation

$(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$ is y =

(Multan Board 2018 G-I)

(a) $\log_a (e^x + e^{-x}) + C$ (b) $\log_e (e^x + e^{-x}) + C$

(c) $\log_a (e^x - e^{-x}) + C$ (d) $\log_e (e^x - e^{-x}) + C$

Q.28 Solution of Differential Equation $\frac{dy}{dx} = Sec^2x$ is:

(Bahawalpur Board 2019)

(a) $y = Cot\, x + c$ (b) $y = \tan x + c$

(c) $y = Cos\, x + c$ (d) $y = -\tan x + c$

Q.29 Order of the differential equation

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2x = 0$ is

(Gujranwala Board 2019 G-II)

(a) 0 (b) 1

(c) 2 (d) 3

Q.30 The order of differential equation

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 3x = 0$ is: *(Rawalpindi Board 2019)*

(a) 2 (b) 1

(c) 0 (d) 3

Q.31 The order of differential equation

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2 = 0$ is: *(Sahiwal Board 2019)*

(a) 1 (b) 2

(c) 3 (d) 4

Chapter 4

INTRODUCTION TO ANALYTICAL GEOMETRY

EXERCISE 4.1

SHORT ANSWERS TO THE QUESTIONS

Q.5 Find the distance between A(–8,3) and B(2,–1).
(Rawalpindi Board 2013)(Sahiwal Board 2013)

Ans. Given A(–8,3), B(2,–1)

$$|AB| = \sqrt{(-8-2)^2 + (3-(-1))^2}$$
$$|AB| = \sqrt{100+16}$$
$$|AB| = \sqrt{116} = 2\sqrt{29}$$

Q.9 Show that the points A(0, 2), $B(\sqrt{3}, -1)$ and C(0, –2) are vertices of a right triangle.
(D.G Khan Board 2017 G-II)(Sargodha Board 2019)
(Bahawalpur Board 2019)

Ans. $|AB|^2 = (\sqrt{3}-0)^2 + (-1-2)^2$

$= (\sqrt{3})^2 + (-3)^2 = 3 + 9 = 12$ ..... (1)

$|BC|^2 = (0-\sqrt{3})^2 + (-2-(-1))^2$

$= (-\sqrt{3})^2 + (-2+1)^2$

$= (\sqrt{3})^2 + (-1)^2 = 3 + 1 = 4$ ..... (2)

$|AC|^2 = (0-0)^2 + (-2-2)^2$

$= 0 + (-4)^2 = 16$ ..... (3)

From (1), (2), (3) we get

$|AB|^2 + |BC|^2 = |AC|^2$

i.e., Pythagoras theorem is satisfied.

Thus the points A, B, C are vertices of a right triangle.

Q.13 Find mid point of the line segment joining the points A(3, 1), B(–2, –4).
(Bahawalpur Board 2014)(Faisalabad Board 2017)

Ans. Mid point of $\overline{AB} = \left(\frac{3+(-2)}{2}, \frac{1+(-4)}{2}\right)$

$= \left(\frac{3-2}{2}, \frac{1-4}{2}\right)$

$= \left(\frac{1}{2}, \frac{-3}{2}\right)$

Q.14 Find the mid point of the line joining the two points A(–8, 3) , B(2, 1). *(Lahore Board 2017G-I)*

Ans. Mid point $= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$= \left(\frac{-8+2}{2}, \frac{3+1}{2}\right) = (-3, 2)$

Q.16 The points A(–5, –2) and B(5, –4) are ends of a diameter of a circle. Find its Centre and Radius.
(Bahawalpur Board 2016(Sahiwal Board 2019))
(Multan Board 2017 G-I)(Lahore Board 2017 G-I)

Ans. The centre of the circle is the mid point of diameter $\overline{AB}$ i.e.,

$\left(\frac{-5+5}{2}, \frac{-2+(-4)}{2}\right) = \left(\frac{0}{2}, \frac{-6}{2}\right) = (0, -3)$

Radius of the circle is half of the length of diameter of the circle

i.e. Radius of the circle $= \frac{1}{2}|AB|$

$= \frac{1}{2}\sqrt{(5-(-5))^2 + (-4-(-2))^2}$

$= \frac{1}{2}\sqrt{(5+5)^2 + (-4+2)^2}$

$= \frac{1}{2}\sqrt{(10)^2 + (-2)^2} = \frac{1}{2}\sqrt{100+4}$

$= \frac{1}{2}\sqrt{104} = \frac{1}{2}\sqrt{4\times 26}$

$= \frac{1}{2}(2\sqrt{26}) = \sqrt{26}$

Q.18 Find the coordinates of the point that divides the join of A(–6, 3) and B(5, –2) in the ratio 2 : 3 internally. *(Multan Board 2017 G-I)*
(Lahore Board 2015, 2019 G-II)
(Gujranwala Board 2012, 2019 G-II, 2018)
(Rawalpindi Board 2017 G-II)(Bahawalpur Board, 2018)

Ans. Here $k_1 = 2, k_2 = 3, x_1 = -6, x_2 = 5$

By the formula, we have

$$x = \frac{2\times 5 + 3\times(-6)}{2+3} = \frac{-8}{5} \text{ and } y = \frac{2(-2)+3(3)}{2+3} = 1$$

Coordinates of the required point are $\left(\frac{-8}{5}, 1\right)$.

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 In a plane, two mutually perpendicular lines are called: *(Faisalabad Board 2013)*
(a) Co-ordinate axis (b) Radii
(c) Medians (d) Altitudes

Q.2 The distance of the point (1, 1) from origin is: *(Sargodha Board 2013)*
(a) 0 (b) 1
(c) $\sqrt{2}$ (d) 4

Q.3 Distance between (1, 2) and (2, 1) is: *(D.G.K Board 2014 G-I, Multan Board 2015 G-I) (Lahore Board 2016 G-I)*
(a) $\sqrt{3}$ (b) $\sqrt{5}$
(c) $\sqrt{2}$ (d) 7

Q.4 The co-ordinate axis divide the plane into ——— equal parts: *(D.G.K Board 2014 G-II)*
(a) 2 (b) 3
(c) 4 (d) 5

Q.5 Distance between the points (2, 3) and (3, 2) is: *(D.G.K Board 2015 G-I)*
(a) 2 (b) $\sqrt{2}$
(c) 1 (d) $2\sqrt{5}$

Q.6 Mid-point of the line segment joining the points (3, −1) and (−3, 1) is: *(D.G.K Board 2013 G-II)*
(a) (0, 0) (b) (1, 1)
(c) (2, 2) (d) (3, 3)

Q.7 The distance of the point (a, b) from origin is: *(D.G.K Board 2013 G-II)*
(a) $\sqrt{a-b}$ (b) $\sqrt{a+b}$
(c) $\sqrt{a^2-b^2}$ (d) $\sqrt{a^2+b^2}$

Q.8 The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is: *(Multan Board 2015 G-II)*
(a) $(x_2-x_1)^2+(y_2-y_1)^2$
(b) $\sqrt{(x_2-x_1)+(y_2-y_1)}$
(c) $\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$
(d) $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Q.9 For any point (x, y) on x-axis. *(Lahore Board 2014 G-II)*
(a) y = 0 (b) y = −1
(c) y = 1 (d) y = 2

Q.10 Location in the plane of the point P(x, y) for which x = 0 is: *(Lahore Board 2012 G-II)*
(a) x-axis (b) y-axis
(c) 1st quadrant (d) 2nd quadrant

Q.11 Mid point of A(2, 0), B(0, 2) is: *(Bahawalpur Board 2016)*
(a) (0, 2) (b) (2, 0)
(c) (2, 2) (d) (1, 1)

Q.12 In a plane two mutually perpendicular lines x′ox and y′oy, one horizontal and the other vertical are called.
(a) X-axis (b) Coordinate axes
(c) Y-axis (d) None of these

Q.13 If (x, y) are the coordinate of a point P, then the second component of the ordered pair is called.
(a) Abscissa (b) Ordinate
(c) Coordinate (d) Ordered pair

Q.14 x is negative and y is positive in
(a) First quadrant (b) Second quadrant
(c) Third quadrant (d) Fourth quadrant

Q.15 x is positive and y is negative in:
(a) First quadrant (b) Second quadrant
(c) Third quadrant (d) Fourth quadrant

Q.16 x and y both are positive in:
(a) First quadrant (b) Second quadrant
(c) Third quadrant (d) Fourth quadrant

Q.17 The horizontal line x′ox is called.
(a) X-axis (b) Y-axis
(c) Ordinate (d) Abscissa

Q.18 The vertical line y′oy is called.
(a) X-axis (b) Y-axis
(c) Abscissa (d) Ordinate

Q.19 If diagonals of a parallelogram are equal then it is a:
(a) Square (b) rhombus
(c) Rectangle (d) None of these

Q.20 If a pair of opposite sides of a quadrilateral are equal and parallel then it is a:
(a) Rectangle (b) Rhombus
(c) Parallelogram (d) None of these

Q.21 The lines lying on the same plane are called.
(a) Collinear lines (b) Coplanar lines
(c) Concurrent (d) Parallel lines

Q.22 A quadrilateral having two parallels and two non-parallel sides is called.
(a) Trapezium (b) Rectangle
(c) Rhombus (d) None of these

Q.23 The distance between the points (0, 0) and (1, 2) is. *(Sahiwal Board 2017)(Lahore Board 2017 G-I)*
(a) 0 (b) 2
(c) $\sqrt{3}$ (d) $\sqrt{5}$

Q.24 If (3, 5) is the mid point of (5, a) and (b, 7), then
(Multan Board 2017 G-I)
(a) a = 4 , b = 2 (b) a = 3 , b = 3
(c) a = 7 , b = 2 (d) a = 3 , b = 1

Q.25 The distance of point (−1, 3) from x-axis is.
(Multan Board 2017 G-I)
(a) −1 (b) 3
(c) 2 (d) −4

Q.26 The point (3, −8) lies in the quadrant.
(Rawalpindi Board 2017 G-II)
(a) I (b) II
(c) III (d) IV

Q.27 The x-component of a point P(x, y) is called.
(Sahiwal Board 2017)
(a) Ordinate (b) Abscissa
(c) Coordinate (d) Distance from origin

Q.28 The y-component of a point P(x, y) is called
(a) Ordinate (b) Abscissa
(c) Coordinate (d) Distance from origin

Q.29 Centroid of triangle with vertices A(2, 1), B(−1, 3), and C(−1, −4) is: *(Lahore Board 2012 G-II)*
(Sargodha Board 2018)
(a) (3, 1) (b) (0, 0)
(c) (2, 2) (d) (−2, −5)

Q.30 Centroid of a triangle with vertices A(2, 3), B(−1, 6) and C(7, 5) is: *(Lahore Board 2013 G-II)*
(a) $\left(\frac{1}{3}, \frac{2}{7}\right)$ (b) $\left(\frac{2}{3}, \frac{7}{3}\right)$
(c) $\left(\frac{8}{3}, \frac{14}{3}\right)$ (d) $\left(\frac{7}{3}, \frac{11}{3}\right)$

Q.31 Distance of the point (−2 , 3) from y-axis is:
(Faisalabad Board 2019 G-II)
(a) 2 (b) − 2
(c) 3 (d) − 3

Q.32 Centroid of triangle with vertices A(2, 1), B(−1, 3) and C(−1, −4) is
(Gujranwala Board 2019 G-I)
(a) (3, 1) (b) (0, 0)
(c) (2, 2) (d) (−2, −5)

Q.33 Distance between (1, 2) and (2, 1) is
(Gujranwala Board 2019 G-I)
(a) $\sqrt{3}$ (b) $\sqrt{5}$
(c) $\sqrt{2}$ (d) 7

Q.34 The vertices of a triangle are (a, b - c), (b, c - a)(e, a − b) then its centroid is
(Gujranwala Board 2019 G-II)
(a) $\left(0 + \frac{a+b+c}{3}\right)$ (b) $\left(0 + \frac{a-b-c}{3}\right)$
(c) (0 , 0) (d) $\left(\frac{a+b+c}{3}, 0\right)$

Q.35 The mid point of line segment joining A(−8, 3), B(2, − 1) is: *(Lahore Board 2019)*
(a) (−6 , 2) (b) (−3 , 1)
(c) (1 , 4) (d) (1 , 5)

Q.36 If (4 , −2) , (−2 , 4) and (4, 10) are vertices of triangle, then its centroid will be
(Sargodha Board 2019)
(a) (−2 , 4) (b) (2 , 4)
(c) (2 , −4) (d) (−2 , −4)

Q.37 The distance between two points A(-8 , 3), B(2 , -1) is: *(Faisalabad Board 2019)*
(a) 116 (b) (−6 , 2)
(c) $2\sqrt{29}$ (d) $\sqrt{58}$

EXERCISE 4.2

SHORT ANSWERS TO THE QUESTIONS

Q.1 The two points P and O′ are given in xy co-ordinate system. Find XY-co-ordinate of Preferred to the co-ordinate axis O′X and O′Y, where P(−2,6) and O′(−3,2).
(Rawalpindi Board 2013)(Multan Board 2018 G-II)

Ans. Given P(x, y) = P(−2, 6) ⇒ x = −2, y = 6
O′ (h, h) = O′ (−3, 2) ⇒ h = −3, k = 2
As we know.

X = x − h	Y = y − h
X = −2 + 3	Y = 6 − 2
X = 1	Y = 4

So, P(X, Y) = P(1, 4)

Q.2 The two points P and O′ are given in xy-coordinate system. Find the XY-coordinates of P referred to O′X and O′Y where P (−6, −8); O′(−4, −6)
(Gujranwala Board 2012)

Ans. Here (x, y) = (−6, −8) , (h, k) = (−4, −6)
Coordinates of P referred to XY-coordinate system are
X = x − h = −6 − (−4) = −6 + 4 = −2
Y = y − k = −8 − (−6) = −8 + 6 = −2
Hence P(X, Y) = P (−2, −2)

Q.3 The coordinates of a point P are (–6, 9). The axes are translated through the point O′(–3, 2). Find the coordinates of P referred to the new axes. *(Faisalabad Board 2016)*

Ans. Here h = –3, k = 2

Coordinates of P referred to the new axes are (X, Y) given by

X = –6 – (–3) = –3 and Y = 9 – 2 = 7

Thus P(X, Y) = P(–3, 7).

Q.4 The xy-coordinate axes are rotated about the origin through the indicated angle. The new axes are OX and OY. Find the XY–cordinates of the point P with the given xy-coordinates: P(5,3) and θ = 45°.

(Multan Board 2013)(Lahore Board 2012 G-I)

(Gujranwala Board 2019 G-II)

Ans. Here x = 5, y = 3, and θ = 45°

We are to find P(x,y)

As $X = x\cos\theta + y\sin\theta \Rightarrow x = 5\cos 45° + 3\sin 45°$

OR $x = 5.\frac{1}{\sqrt{2}} + 3.\frac{1}{\sqrt{2}} = \frac{8}{\sqrt{2}} = 4 \times \frac{2}{\sqrt{2}} = 4\sqrt{2}$

$Y = y\cos\theta - x\sin\theta = 3\cos 45° - 5\sin 45°$

OR $y = 3.\frac{1}{\sqrt{2}} - 5\frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$

Thus $P(x,y) = (4\sqrt{2}, -\sqrt{2})$ Ans

Q.5 Find XY-coordinates of P obtained by rotating xy-coordinates through an angle θ where P(3, –7) ; θ = 30°

(Lahore Board 2011 G-II, 2012 G-II)

Ans. Let P(X, Y) be the coordinates of P referred to the XY axes.

Angle of rotation is given by θ = 30°

$\therefore \quad X = x\cos\theta + y\sin\theta$

$= 3\cos 30° + (-7)\sin 30° = 3\left(\frac{\sqrt{3}}{2}\right) - 7\left(\frac{1}{2}\right)$

$= \frac{3\sqrt{3} - 7}{2}$

$Y = -x\sin\theta + y\cos\theta$

$= -3\sin 30° + (-7)\cos 30°$

$= -3\left(\frac{1}{2}\right) - 7\left(\frac{\sqrt{3}}{2}\right) = \frac{-3 - 7\sqrt{3}}{2}$

Thus $P(X, Y) = \left(\frac{3\sqrt{3} - 7}{2}, \frac{-3 - 7\sqrt{3}}{2}\right)$

Q.6 The xy-coordinates axes are rotated about the origin through an angle of 30°. If the xy-coordinates axes of a point are (5, 7), find its XY-coordinates, where OX and OY are the axes obtained after rotation.

(Gujranwala Board 2010)(Rawalpindi Board 2014)

Ans. Let (X, Y) be the coordinates of P referred to the XY-axes.

Here θ = 30°, x = 5, y = 7

From equations $X = x\cos\theta + y\sin\theta$

$Y = y\cos\theta - x\sin\theta$, we have

$X = 5\cos 30° + 7\sin 30°$ and

$Y = -5\sin 30° + 7\cos 30°$

or $X = \frac{5\sqrt{3}}{2} + \frac{7}{2}$ and $Y = \frac{-5}{2} + \frac{7\sqrt{3}}{2}$

i.e., $(X,Y) = \left(\frac{5\sqrt{3}+7}{2}, \frac{-5+7\sqrt{3}}{2}\right)$

are the required coordinates.

Q.7 The xy coordinate axes are rotated through angle θ = 30° and axes are OX & OY. Find x, y coordinates of P with. P(X, Y) = (–5, 3)

(Faisalabad Board 2013)

Ans. P(x, y)= (–5, 3)

X = –5, Y = 3

θ = 30°

We know that

$x = X\cos\theta - Y\sin\theta$

$y = X\sin\theta + Y\cos\theta$

$x = -5\frac{\sqrt{3}}{2} - 3\frac{1}{2} = \frac{-5\sqrt{3} - 3}{2}$

$y = -5\frac{1}{2} + 2 \times \frac{\sqrt{3}}{2}$

$= \frac{-5 + 3\sqrt{3}}{2}$

So, $P(x,y) = P\left(\frac{3 - 5\sqrt{3}}{2}, \frac{3\sqrt{3} + 5}{2}\right)$

MULTIPLE CHOICE QUESTIONS

Q.1 The co-ordinates of a point P(x, y) translated through the point O′ (h, k), then the co-ordinates of P referred to new axis are.

(Bahawalpur Board 2018)

(a) (x – h , y – k) (b) (x + h , y + k)

(c) (x – k , y – h) (d) (x + k , y + h)

EXERCISE 4.3

SHORT ANSWERS TO THE QUESTIONS

Q.1 By means of slopes, show that the points (4, – 5) ; (7, 5) ; (10, 15) lie on a line. *(Sargodha Board 2017)*

Ans. Let A(4, –5); B (7, 5); C (10, 15) be the given points.

Slope of $\overline{AB} = \frac{5-(-5)}{7-4} = \frac{5+5}{3} = \frac{10}{3}$

Slope of $\overline{BC} = \frac{15-5}{10-7} = \frac{10}{3}$

Slope of $\overline{AB}$ = Slope of $\overline{BC}$

Hence three points A(4, –5); B(7, 5); C(10, 15) lie on a line.

Q.2 By means of slopes, show that the following points lie on a line: (–1, –3) ; (1, 5) ; (2, 9) *(Lahore Board 2017 G-I, 2019 G-II) (Gujranwala Board 2018)(D.G Khan Board 2017 G-I)*

Ans. A(– 1, – 3) ; B(1, 5) ; C(2, 9) be the given points.

Slope of $\overline{AB} = \frac{5-(-3)}{1-(-1)} = \frac{5+3}{1+1} = \frac{8}{2} = 4$

Slope of $\overline{BC} = \frac{9-5}{2-1} = \frac{4}{1} = 4$

Slope of $\overline{AB}$ = Slope of $\overline{BC}$

Hence three points A, B, C lie on a line.

Q.3 Find k so that the line joining A(7, 3) ; B(k,–6) and the line joining C(–4, 5); D(–6, 4) are perpendicular.

Ans. Slope of line $\overleftrightarrow{AB} = \frac{-6-3}{k-7} = \frac{-9}{k-7}$

Slope of line $\overleftrightarrow{CD} = \frac{4-5}{-6-(-4)} = \frac{4-5}{-6+4}$

$= \frac{-1}{-2} = \frac{1}{2}$

Line $\overleftrightarrow{AB}$ will be perpendicular to line $\overleftrightarrow{CD}$, if

$\left(\text{Slope of } \overleftrightarrow{AB}\right)\left(\text{Slope of } \overleftrightarrow{CD}\right) = -1$

$\Rightarrow \left(-\frac{9}{k-7}\right)\left(\frac{1}{2}\right) = -1 \Rightarrow 9 = 2(k-7)$

$\Rightarrow 2k - 14 = 9$

$\Rightarrow 2k = 14 + 9 \Rightarrow 2k = 23 \Rightarrow k = \frac{23}{2}$

Q.4 The three points A(7, –1), B(–2, 2) and C(1, 4) are consecutive vertices of a parallelogram. Find the fourth vertex. *(Rawalpindi Board 2016)*

Ans. See Long Question 2

Q.5 Find an equation of line through (8, –3) having slope 0. *(Sahiwal Board 2013)*

Ans. Given points is (8, –3)

Slope = m = 0

Equation of line

$y - y_1 = m(x - x_1)$

$y - (-3) = 0(x - 8)$

$y + 3 = 0$ Required line $y = -3$

Q.6 Find an equation of the line through A(–6,5) having slope 7. *(Faisalabad Board 2019 G-I) (Sargodha Board 2013)(Lahore Board 2014 G-I) (Rawalpindi Board 2017 G-II)(Sargodha Board 2017) (Gujranwala Board 2019)*

Ans. Given point is A(–6,5)

Slope = m = y

Equation of required line

$y - y_1 = m(x - x_1)$

$y - 5 = 7(x - (-6))$

$y - 5 = 7x + 42$

$7x + y + 47 = 0$ required equation.

Q.7 Find an equation of the line through (–5, –3) and (9, –1) *(Multan Board 2018) (Faisalabad Board 2019 G-II)*

Ans. Slope of the line is $\frac{-1-(-3)}{9-(-5)} = \frac{-1+3}{9+5}$

$= \frac{2}{14} = \frac{1}{7}$

Using point-slope form equation of the line through

(–5, –3) having slope $\frac{1}{7}$ is $y - (-3) = \frac{1}{7}(x - (-5))$

[applying $y - y_1 = m(x - x_1)$]

$7(y + 3) = x + 5$

$7y + 21 = x + 5$

$x - 7y - 16 = 0$ is the required line.

Q.8 Find an equation of the line having y-intercept – 7 and slope –5 *(Sahiwal Board 2014 G-II)(Multan Board 2017 G-I)*

Ans. Here $(x_1, y_1) = (0, -7)$ and $m = -5$

Using point-slope form equation of the line through (0, –7) having slope –5 is

$y - (-7) = -5(x - 0)$

[applying $y - y_1 = m(x - x_1)$]

$y + 7 = -5x$

$5x + y + 7 = 0$ is the required line.

Q.9 Find an equation of the line through (–4, –6) and perpendicular to a line having slope $-\frac{3}{2}$.

(Lahore Board 2015 G-II)(Rawalpindi Board 2013)

Ans. Here $(x_1, y_1) = (-4, -6)$

We know that product of slopes of two perpendicular lines is –1.

$\therefore$ m = Slope of required line $= -\frac{1}{-3/2} = \frac{2}{3}$

Required line using point slope form is

$y - y_1 = m(x - x_1)$

$y - (-6) = \frac{2}{3}(x - (-4))$

$3(y + 6) = 2(x + 4)$

$\Rightarrow \quad 3y + 18 = 2x + 8$

$2x - 3y + 8 - 18 = 0$

$\Rightarrow \quad 2x - 3y - 10 = 0$

Q.10 Convert the equation 4x + 7y – 2 = 0 into two intercept form. ***(Lahore Board 2017 G-I,II)***

Ans. $47x + 7y - 2 = 0$

$4x + 7y = 2$

$\frac{4x}{2} + \frac{7y}{2} = \frac{2}{2}$

$2x + \frac{7y}{2} = 1$

$\frac{x}{\frac{1}{2}} + \frac{y}{\frac{2}{7}} = 1$

Compare it with

$\frac{x}{a} + \frac{y}{b} = 1$

$a = \frac{1}{2}, \; b = \frac{2}{7}$

Q.11 Convert the equation 15y – 8x + 3 = 0 into two intercept form. ***(Gujranwala Board 2017)***

Ans. $15y - 8x + 3 = 0$

$8x - 15y = 3$

Dividing by 3

$\frac{8x}{3} - \frac{15y}{3} = \frac{3}{3}$

$\frac{x}{\frac{3}{8}} + \frac{y}{\frac{-1}{5}} = 1$

Compare it with $\frac{x}{a} + \frac{y}{b} = 1$

$a = \frac{3}{8}, \; b = -\frac{1}{5}$

Q.12 Check whether the two lines 4y + 2x – 1 = 0, x – 2y – 7=0 are (i) Parallel (ii) Perpendicular

(D.G Khan Board 2014 G-II)

Ans. $4y + 2x - 1 = 0$ .....(1)

$x - 2y - 7 = 0$ .....(2)

Slope of line (1) $= -\frac{\text{coeff. of } x}{\text{coeff. of } y} = -\frac{2}{4} = -\frac{1}{2}$

Slope of line (2) $= -\frac{\text{coeff. of } x}{\text{coeff. of } y} = -\frac{1}{-2} = \frac{1}{2}$

Hence lines are neither parallel nor perpendicular.

Q.13 Check whether the two lines $2x + y - 3 = 0, 4x + 2y + 5 = 0$ are (i) Parallel (ii) Perpendicular

Ans. $2x + y - 3 = 0$ .....(1)

$4x + 2y + 5 = 0$ .....(2)

Slope of line (1) $= -\frac{\text{coeff. of } x}{\text{coeff. of } y} = -\frac{2}{1} = -2$

Slope of line (2) $= -\frac{\text{coeff. of } x}{\text{coeff. of } y} = -\frac{4}{2} = -2$

Slopes are equal, hence given pair of lines are parallel.

Q.14 Find an equation of the line through (5, –8) and perpendicular to the join of A(–15, –8), B(10, 7)

(Lahore Board 2015 G-I)

(Multan Board 2017 G-I)(Gujranwala Board 2018)

Ans. Slope of the $\overline{AB} = \frac{7 - (-8)}{10 - (-15)} = \frac{7 + 8}{10 + 15}$

$= \frac{15}{25} = \frac{3}{5}$

Slopes of a line perpendicular to the $\overline{AB}$ is

$-\frac{1}{3/5} = -\frac{5}{3}$

Hence equation of required line using point slope form is

$y - (-8) = -\frac{5}{3}(x - 5)$

$3(y + 8) = -5(x - 5) \Rightarrow 3y + 24 = -5x + 25$

$\Rightarrow 5x + 3y + 24 - 25 = 0$

$\Rightarrow 5x + 3y - 1 = 0$

Q.15 Find whether point (5, 8) lies above or below the line 2x – 3y + 6 = 0. ***(Sahiwal Board 2019)***

(Faisalabad Board 2013) (Sargodha Board 2013)

Ans. Given equation is

$2x - 3y + 6 = 0$

Make coefficient of y + ve

$3y - 2x - 6 = 0$

Now put (5, 8) in L.H.S of equation (i)

L.H.S = $3y - 2x - 6$

$= 3(8) - 2(5) - 6$

$= 24 - 10 - 6$

$= 8 = +ve$

$\therefore$ (5, 8) lies above the given line.

Q.16 Chech whether the given point lies above or below the given line. P(–7, 6); 4x + 3y – 9 = 0

(Sargodha Board 2016)(Bahawalpur Board 2018)

Ans. $4x + 3y - 9 = 0$ ..... (1)

Putting (– 7 , 6) in the L. H.S. of (1), we have

$4(-7) + 3(6) - 9 = -28 + 18 - 9 = 18 - 37$

$= -19$ ..... (2)

Thus the coefficient of y in (1) and the expression (2) have opposite signs and so the point (–7, 6) lies below the line (1).

Q.17 Find the distance from the point P(6, –1) to the line 6x – 4y + 9 = 0

(D.G Khan Board 2014 G-II)(Bahawalpur Board 2016)

(Sargodha Board 2017, 2018)(Multan Board 2014 G-II)

(Multan Board 2017 G-I, 2018 G-II)

(A.J.K board 2017)(Gujranwala Board 2018)

Ans. By using the formula $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$, the required distance is

$$d = \frac{|6(6) - 4(-1) + 9|}{\sqrt{6^2 + (-4)^2}} = \frac{|36 + 4 + 9|}{\sqrt{36 + 16}} = \frac{49}{\sqrt{52}}$$

Q.18 Find the area of the region bounded by the triangle with vertices (a, b + c), (a, b – c) and (–a, c). ***(D.G Khan Board 2017 G-I)***

(Sahiwal Board 2018)

Ans. Let A(a, b + c) , B(a, b – c) and C(–a, c), then

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ a & b-c & 1 \\ -a & c & 1 \end{vmatrix}$$

$$= \frac{1}{2}\,[a(b - c - c) - (b - c)(a + a) + 1(ac + ab - ac)]$$

$$= \frac{1}{2}\,[a(b - 2c) - (b - c)(2a) + ab]$$

$$= \frac{1}{2}\,[ab - 2ac - 3ab + 3ac + ab] = \frac{1}{2}\,[0] = 0$$

LONG QUESTIONS

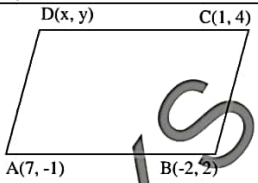
Q.1 By means of slopes, show that the points lie on the same line. (–1, –3), (1, 5),(2, 9).

(Sahiwal Board 2015)

Sol: See Short Question 15

Q.2 The three points A(7, –1), B(–2, 2) and C(1, 4) are consecutive vertices of a parallelogram, find the fourth vertex.

Sol: Let D(x, y) be the fourth vertex



Slope of AB $= m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 + 1}{-2 - 7} = \frac{3}{-9} = \frac{-1}{3}$

Slope of BC $= m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{1 + 2} = \frac{2}{3}$

Slope of CD $= m_3 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - 4}{x - 1}$

Slope of DA $= m_4 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - y}{7 - x}$

As ABCD is a parallelogram

So $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$

$\Rightarrow$ $m_1 = m_3$ and $m_2 = m_4$

$$-\frac{1}{3} = \frac{y - 4}{x - 1} \qquad \left|\ \frac{2}{3} = \frac{-1 - y}{7 - x}\right.$$

$\Rightarrow -x + 1 = 3y - 12$ $\quad\left|\ \Rightarrow 14 - 2x = -3 - 3y\right.$

$\Rightarrow x + 3y = 13$ ... (1) $\quad\left|\ \Rightarrow 2x - 3y = 17 \text{ ... (2)}\right.$

adding eq (1) and (2)

$3x = 30 \quad \Rightarrow x = 10$

(1) $\Rightarrow 10 + 3y = 13 \Rightarrow 3y = 3 \Rightarrow y = 1$

Hence D (x, y) = (10, 1)

Q.3 Find an equation of the line through (5, –8) and perpendicular to the join of A(–15, –8), B(10, 7).

(Faisalabad Board 2013)

Sol: See Short Question 73

Q.4 Find equations of two parallel lines perpendicular to 2x – y + 3 = 0 such that the product of the x and y intercept of each is 3.

(Lahore Board 2012 G-II, 2015 G-I, 2019 G-II)

(D.G.K Board 2014)(Rawalpindi Board 2017)

(Faisalabad Board 2019 G-II)

Sol: Given line is $2x - y + 3 = 0$

Any line perpendicular to given line is

$x + 2y + c = 0$ ..... (1)

For x – intercept put $y = 0$

$x + c = 0 \quad \Rightarrow \quad \boxed{x = -c}$

For y – intercept put $x = 0$

$$2y + c = 0 \quad \Rightarrow \quad \boxed{y = \frac{-c}{2}}$$

According to given condition

$$(x - \text{intercept})\ (y - \text{intercept}) = 3$$

$$(-c)\left(\frac{-c}{2}\right) = 3$$

$$\Rightarrow \quad c^2 = 6 \quad \Rightarrow \quad c = \pm\sqrt{6}$$

Putting $c = \pm\sqrt{6}$ in (i)

$$x + 2y \pm \sqrt{6} = 0$$

Which are required equations of line.

Q.5 Find distance between $3x - 4y + 3 = 0$ and $3x - 7y + 7 = 0$. Also find equation of parallel line lying midway between them.

(Lahore Board 2016 G-I)

Sol: $\lambda_1 : 3x - 4y + 3 = 0$ ..... (i)

$\lambda_2 : 3x - 4y + 7 = 0$ ..... (ii)

Put $x = 0$ in eq (1)

$0 - 4y + 3 = 0$

$y = \frac{3}{4} \Rightarrow A\left(0, \frac{3}{4}\right)$

Put $y = 0$ in eq (1)

$3x - 0 + 3 = 0 \Rightarrow 3x = -3$

$x = -1 \Rightarrow$ B (–1, 0)

Distance from $A\left(0, \frac{3}{4}\right)$ to $3x - 4y + 7 = 0$ is

$$d = \frac{\left|3(0) - 4\left(\frac{3}{4}\right) + 7\right|}{\sqrt{(3)^2 + (-4)^2}} = \frac{|0 - 3 + 7|}{\sqrt{9 + 16}} = \frac{|4|}{\sqrt{25}} = \frac{4}{5}$$

put $x = 0$ in eq (2)

$0 - 4y + 7 = 0 \Rightarrow y = \frac{7}{4}$

$\Rightarrow C\left(0, \frac{7}{4}\right)$

Put $y = 0$ in eq (2)

$3x - 0 + 7 = 0 \Rightarrow 3x = -7$

$x = \frac{-7}{3} \Rightarrow D = \left(\frac{-7}{3}, 0\right)$

mid point of AC $\left(\frac{0+0}{2}, \frac{\frac{3}{4}+\frac{7}{4}}{2}\right) = \left(0, \frac{5}{4}\right)$

(Θ Mid point $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$)

Slope of a line $\lambda_1 : 3x - 4y + 3 = 0$ is $\frac{3}{4}$

(Slope of a line $= -\frac{a}{b}$)

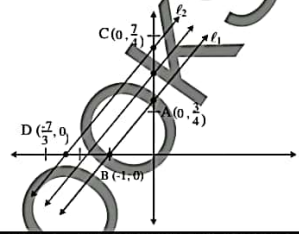
Slope of required line $= m = \frac{3}{4}$ (Θ lines are Parallel)

Equation of a line passing through $\left(0, \frac{5}{4}\right)$ and having slope $\frac{3}{4}$

$y - y_1 = m(x - x_1) \Rightarrow y - \frac{5}{4} = \frac{3}{4}(x - 0)$

Multiplying both sides by 4

$4y - 5 = 3x \quad \Rightarrow \quad 3x - 4y + 5 = 0$



MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 Slope of a straight line $ax + by + c = 0$ is:

(Faisalabad Board 2013)

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$

(c) $-\frac{a}{b}$ (d) $\frac{c}{a}$

Q.2 The two intercept form of straight line is:

(Rawalpindi Board 2013)(D.G. Khan Board 2013 G-I)
(Multan Board 2015 G-I)(A.J.K Board 2017)

(a) $y = mx + C$ (b) $\frac{x}{a} + \frac{y}{b} = 1$

(c) $y - y_1 = m(x - x_1)$ (d) $x\cos\alpha + y\sin\alpha = p$

Q.3 The point $p(x_1, y_1)$ and the origin are on the same side of $ax + by + C = 0$.

If $ax_1 + by_1 + C$ and C have the

(Sargodha Board 2013)

(a) $c = 0$ (b) Same sign

(c) Opposite sign (d) Does not possible

Q.4 If A(–3, 6) and B(3, 2), then slope of AB is:

(Multan Board 2013 G-I)

(a) $\frac{3}{2}$ (b) $-\frac{2}{3}$

(c) $\frac{1}{3}$ (d) $-\frac{3}{2}$

Q.5 Slope-intercept form of straight line is:
(Multan Board 2011 G-I, Sahiwal Board 2013 G-I)
(a) $y = mx + C$ (b) $x\cos\alpha + y\sin\alpha = p$
(c) $\frac{x}{a} + \frac{y}{b} = 1$ (d) $x = 0$

Q.6 Equation of line in the form of $x\cos\alpha + y\sin\alpha = p$ is: *(Multan Board 2013 G-II)*
(a) Symmetric form (b) Intercept form
(c) Slope-intercept (d) Normal form

Q.7 The slope of a line with inclination 90° is:
(Multan Board 2014 G-II, Rawalpindi Board 2014)
(a) 0 (b) $\frac{1}{\sqrt{3}}$
(c) 1 (d) Undefined

Q.8 The lines l_1, l_2 with slopes m_1, m_2 are perpendicular if: *(Rawalpindi Board 2014)*
(Lahore Board 2017 G-I)
(Gujranwala, Bahawalpur, Faisalabad Board 2018)
(a) $m_1m_2 = -1$ (b) $m_1 = m_2$
(c) $m_1 + m_2 = a$ (d) $m_1m_2 = 1$

Q.9 The lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ will be perpendicular if:
(Bahawalpur Board 2014)
(a) $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ (b) $a_1a_2 = b_1b_2$
(c) $a_1a_2 + b_1b_2 = 0$ (d) $a_1c_1 + a_2c_2 = 0$

Q.10 The perpendicular distance of the line $12x+5y=7$ from origin is:
(D.G.K Board 2010, Bahawalpur Board 2014)
(a) 13 (b) $\frac{13}{7}$
(c) $\frac{7}{13}$ (d) $\frac{1}{13}$

Q.11 If two lines with slopes m_1 and m_2 are parallel then. *(Multan Board 2014 G-I, 2018 G-II)*
(Sargodha Board 2015)(D.G Khan Board 2017 G-I)
(a) $m_1m_2 = -1$ (b) $m_1m_2 = 1$
(c) $m_1 + m_2 = 0$ (d) $m_1 - m_2 = 0$

Q.12 If α is the inclination of non vertical lines then slope is: *(D.G.K Board 2014 G-I)*
(a) $\sin\alpha$ (b) $\cos\alpha$
(c) $\tan\alpha$ (d) $\cot\alpha$

Q.13 For $b > 0$, the point (x_1, y_1) is above the line $ax + by + C = 0$ if: *(Sahiwal Board 2014)*
(a) $ax_1 + by_1 + C < 0$ (b) $ax_1 + by_1 + C = 0$
(c) $ax_1 + by_1 + C > 0$ (d) $ax_1 - by_1 - C < 0$

Q.14 The slope of line with inclination 30° is:
(Sahiwal Board 2014)
(a) Zero (b) $\frac{1}{\sqrt{3}}$
(c) 1 (d) $\sqrt{3}$

Q.15 The x-intercept of the line $2x + 3y - 1 = 0$ is:
(D.G.K Board 2015 G-II)
(a) $\frac{1}{2}$ (b) 2
(c) 3 (d) $\frac{1}{3}$

Q.16 What is the nature of the line $ax + by + C = 0$ when $a = 0$ and $b \neq 0$, $C \neq 0$.
(D.G.K Board 2015 G-II)(Faisalabad Board 2016)
(a) Line is parallel to y-axis
(b) Line is parallel to x-axis
(c) Line passes through origin
(d) Line is perpendicular to x-axis

Q.17 Slope of the line through the points (x_1, y_1) and (x_2, y_2) is: *(D.G.K Board 2013 G-I)*
(a) $\frac{x_2 - x_1}{y_2 - y_1}$ (b) $\frac{y_2 - y_1}{x_2 - x_1}$
(c) $\frac{y_2 + y_1}{x_2 + x_1}$ (d) $\frac{y_2 - x_2}{y_1 - x_1}$

Q.18 Equation of horizontal line through (a, b) is:
(D.G.K Board 2012 G-II)(Lahore Board 2017 G-II)
(a) $y = a$ (b) $y = b$
(c) $x = a$ (d) $x = b$

Q.19 Slope of line $-ax + by - c = 0$ is:
(D.G.K Board 2012 G-II)
(a) $\frac{a}{b}$ (b) $-\frac{a}{b}$
(c) $\frac{b}{a}$ (d) $\frac{a}{c}$

Q.20 If $m_1 = m_2$, then both lines will be:
(D.G.K Board 2011)
(a) Perpendicular (b) Parallel
(c) Concurrent (d) Non-coplanar

Q.21 If α is the inclination of non-vertical line "l", then its slope is: *(D.G.K Board 2010)(A.J.K Board 2017)*
(Faisalabad Board 2017)
(a) $\sin\alpha$ (b) $\cos\alpha$
(c) $\tan\alpha$ (d) $\cot\alpha$

Q.22 The perpendicular distance of the line $3x + 4y + 10 = 0$ from origin is: *(D.G.K Board 2010)*
(A.J.K Board 2017)
(a) 2 (b) 1
(c) 0 (d) 3

Q.23 Length of perpendicular from (0, 0) to line $4x - 3y - 1 = 0$ equals: *(Sahiwal Board 2015)*

(a) 3 (b) 4

(c) 5 (d) $\frac{1}{5}$

Q.24 The slope of the line with inclination 0° is: *(Multan Board 2015 G-II)*

(a) 0 (b) $\frac{1}{\sqrt{3}}$

(c) 1 (d) $\sqrt{3}$

Q.25 Vertical line passes through (5, 4) is:

(a) $y = 4$ (b) $x = 5$

(c) $y = 5$ (d) $y = -4$

Q.26 Equation of a straight line passing through p(a, b) and parallel to x-axis is: *(Multan Board 2012 G-I)*

(a) $x = a$ (b) $y = b$

(c) $x = b$ (d) $y = a$

Q.27 If α is the inclinator of line "l" $\frac{y - y_1}{\sin\alpha} = \frac{x - x_1}{\cos\alpha}$ is called: *(Lahore Board 2018 G-I)*
(Multan Board 2012 G-I)(D.G. Khan Board 2017 G-I)

(a) Intercept form (b) Point slope form

(c) Symmetric form (d) Normal form

Q.28 Slope of line $3x - 4y + 5 = 0$ equals: *(Multan Board 2011 G-II)*

(a) $\frac{4}{3}$ (b) $\frac{3}{4}$

(c) $-\frac{4}{3}$ (d) $-\frac{3}{4}$

Q.29 The slope of horizontal line equals: *(Multan Board 2011 G-II)*

(a) 1 (b) 2

(c) ∞ (d) 0

Q.30 The slope of line perpendicular to $2x - 3y + 1 = 0$ equals: *(Multan Board 2011 G-I)*

(a) $\frac{2}{3}$ (b) $-\frac{2}{3}$

(c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

Q.31 Which one is equation of straight line perpendicular to x-axis? *(Gujranwala Board 2014)*

(a) $x = k$ (b) $y = k$

(c) $x + y = k$ (d) $x - y = k$

Q.32 Slope of line perpendicular to line $2x - 3y + 1 = 0$ is equal to: *(Lahore Board 2014 G-I)*

(a) $\frac{3}{2}$ (b) $\frac{-3}{2}$

(c) $\frac{2}{3}$ (d) $\frac{-2}{3}$

Q.33 Slope intercept form of line equals. *(Lahore Board 2015 G-II)*

(a) $y - y_1 = m(x - x_1)$ (b) $\frac{x}{a} + \frac{y}{b} = 1$

(c) $x\cos\theta + y\sin\theta = p$ (d) $y = mx + c$

Q.34 Point of intersection of lines $x - 2y + 1 = 0$ and $2x - y + 2 = 0$ equals. *(Lahore Board 2015 G-II)*

(a) (1, 0) (b) (0, 1)

(c) (−1, 0) (d) (0, −1)

Q.35 The distance of point P(6, −1) from the line $6x - 4y + 9 = 0$ is: *(Lahore Board 2015 G-I)*

(a) 49 (b) $\frac{49}{52}$

(c) $\frac{\sqrt{49}}{52}$ (d) $\frac{49}{\sqrt{52}}$

Q.36 The perpendicular distance of the line $3x + 4y + 10 = 0$ from the origin is: *(Gujranwala Board 2013, 2016)*
(Lahore Board 2018 G-I)

(a) 0 (b) 1

(c) 2 (d) 10

Q.37 The distance of point P(x, y) from x-axis is. *(Sargodha Board 2018)*

(a) x (b) y

(c) $|x|$ (d) $|y|$

Q.38 When a line intersects the y-axis at (0, 4), then y-intercept is: *(Lahore Board 2013 G-II)*

(a) 4 (b) 2

(c) 0 (d) 6

Q.39 The ratio in which y-axis divides the line joining (2, −3) and (−5, 6) is: *(Lahore Board 2013 G-I)*

(a) 2 : 3 (b) 2 : 5

(c) 1 : 2 (d) 3 : 5

Q.40 The slope of tangent line to $y = f(x)$ at (x_1, y_1). *(Lahore Board 2013 G-I)*

(a) m (b) $\frac{y_2 - y_1}{x_2 - x_1}$

(c) $f'(x_1)$ (d) $\frac{dy}{dx}$

Q.41 Slope of vertical line is: *(Gujranwala Board 2012)*

(a) $m = 1$ (b) $m = 0$

(c) $m = \infty$ (d) $m = -1$

Q.42 Distance of (−3, 7) from x-axis is: *(Faisalabad Board 2016)*
(Lahore Board 2012 G-I)

(a) 3 (b) −3

(c) 7 (d) 10

Q.43 Slope of y-axis is: *(Bahawalpur Board 2016)*
(a) 0 (b) 1
(c) −1 (d) Undefined

Q.44 Distance of point (1, −2) from y-axis is: *(Bahawalpur Board 2016)*
(a) 0 (b) 1
(c) −1 (d) Undefined

Q.45 The perpendicular distance of $12x + 5y = 7$ from origin is: *(Faisalabad Board 2016, 2017)*
(a) $\frac{1}{13}$ (b) $\frac{7}{13}$
(c) $\frac{13}{7}$ (d) 13

Q.46 Slope of line bisecting 2nd and 4th quadrant is: *(Gujranwala Board 2016)*
(a) $\frac{3\pi}{4}$ (b) 0
(c) −1 (d) 1

Q.47 Equation of the line parallel to $x + 3y - 9 = 0$ is: *(Lahore Board 2016 G-I)*
(a) $3x - y - 9 = 0$ (b) $3x + 9y + 7 = 0$
(c) $2x - 6y - 18 = 0$ (d) $x - 3y + 9 = 0$

Q.48 Slope of line is 1 (one) and angle made by line with x-axis = *(Rawalpindi Board 2016)*
(a) 45° (b) 30°
(c) 60° (d) 75°

Q.49 The distance of the point (3, 7) from x-axis is: *(Sargodha Board 2016)(Multan Board 2018 G-I)*
(a) 3 (b) −3
(c) −7 (d) 7

Q.50 Two lines $a_1x + b_1y + c = 0$ and $a_2x + b_2y + c = 0$ are parallel if: *(Sargodha Board 2016)*
(a) $\frac{a_1}{a_2} = \frac{-b_1}{b_2}$ (b) $\frac{b_1}{c_1} = \frac{b_2}{c_2}$
(c) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ (d) None of these

Q.51 Equation of X-axis is:
(a) $x = 0$ (b) $x = c$
(c) $y = 0$ (d) $y = c$

Q.52 Equation of Y-axis is:
(a) $x = 0$ (b) $y = 0$
(c) $x = c$ (d) $y = c$

Q.53 The line l is vertical if and only if slope is:
(a) 0 (b) 1
(c) 2 (d) Undefined

Q.54 $x = c$ is a line.
(a) Perpendicular to x-axis
(b) Parallel to x-axis
(c) Perpendicular to y-axis
(d) None of these

Q.55 $y = 2x + 3$ is the:
(a) Slope-intercept form (b) Two points form
(c) Pont slope form (d) Intercepts form

Q.56 If the points A, B and C are collinear, then area of the ΔABC will be:
(a) 0 (b) 2
(c) 1 (d) None of these

Q.57 The angle $a \in]0°, 180°[$ measure counter clock wise from positive x-axis to a non horizontal straight line l is:
(a) Slope of l (b) Inclination of l
(c) gradient of l (d) None of these

Q.58 If the line l is parallel to x-axis, then the slope of l is:
(a) 0° (b) 90°
(c) 180° (d) Undefined

Q.59 The line l is horizontal if and only if slope is equal to.
(a) 0 (b) 1
(c) 2 (d) Undefined

Q.60 $y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$ is the
(a) Slope-intercept form (b) Two points form
(c) Pont slope form (d) Intercepts form

Q.61 $\frac{x}{a} + \frac{y}{b} = 1$ is the
(a) Two points form (b) Two intercepts form
(c) Pont-slope form (d) Parametric form

Q.62 $y - y_1 = m(x - x_1)$ is the
(a) Slope-intercept form (b) Point-slope form
(c) Normal form (d) Intercepts form

Q.63 If the point $P(x_1, y_1)$ lies on l, then the distance d would be.
(a) 0 (b) 1
(c) 2 (d) None of these

Q.64 The two lines l_1 and l_2 with respective slopes m_1 and m_2 are parallel if and only if:
(a) $m_1 \neq m_2$ (b) $m_1 \times m_2 = -1$
(c) $m_1 = m_2$ (d) $m_1 \times m_2 \neq -1$

Q.65 $y = c$ a line
(a) Perpendicular to x-axis
(b) Parallel to y-axis
(c) Perpendicular to y-axis
(d) None of these

Q.66 The line $x = a$ is on the right of y-axis, if:
(a) $a > 0$ (b) $a < 0$
(c) $a = 0$ (d) $a \neq 0$

Q.67 Slope of X-axis or of any line parallel to X-axis is:
(a) Zero (b) $\frac{\pi}{2}$
(c) π (d) Undefined

Q.68 Slope of Y-axis or of any line parallel to Y-axis is:
(a) π (b) $\frac{\pi}{2}$
(c) One (d) undefined

Q.69 Equation of a line parallel to X-axis is:
(a) $x = 0$ (b) $y = 0$
(c) $x = c$ (d) $y = k$

Q.70 Equation of a line parallel to Y-axis is:
(a) $x \equiv 0$ (b) $y = 0$
(c) $x = h$ (d) $y = c$

Q.71 Distance of the point (x, y) from X-axis is:
(a) x (b) y
(c) $|x|$ (d) $|y|$

Q.72 The point (x_1, y_1) lies above the line $ax + by + c = 0$ if:
(a) $ax_1 + by_1 + c < 0, b < 0$
(b) $ax_1 + by_1 + c > 0, b < 0$
(c) $ax_1 + by_1 + c = 0, b > 0$
(d) $ax_1 + by_1 + c < 0, b > 0$

Q.73 Angle between two lines whose intercepts on the axes are a, b and a′, b′ respectively is given by:
(a) $\tan^{-1}\left(\frac{ab' - a'b}{aa' + bb'}\right)$ (b) $\tan^{-1}\left(\frac{ab' + a'b}{aa' - bb'}\right)$
(c) $\tan^{-1}\left(\frac{a'b - ab'}{aa' + bb'}\right)$ (d) None of these

Q.74 If the directed distance AP and PB have the same signs, then their ratio is positive and P is said to divide AB.
(a) Internally (b) May be divide
(c) Externally (d) None of these

Q.75 $y = -2$ is a line
(a) Parallel to x-axis (b) Parallel to y-axis
(c) Perpendicular to x-axis (d) None of these

Q.76 The line $y = a$ is above the x-axis, if
(a) $a > 0$ (b) $a < 0$
(c) $a \neq 0$ (d) $a = 0$

Q.77 Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel then
(Faisalabad Board 2017)(Lahore Board 2017 G-II)
(a) $a_1b_2 + a_2b_1 = 0$ (b) $a_1a_2 - b_1b_2 = 0$
(c) $a_1a_2 = b_1b_2 = 0$ (d) $a_1b_2 + a_2b_1 = 0$

Q.78 A parallelogram is a rhombus if and only if its diagonals are
(a) Parallel (b) Perpendicular
(c) Equal (d) None of these

Q.79 The equation of the line that has x-intercept –3 and is perpendicular to the line $3x = 4 - 5y$ is:
(a) $3y \equiv 5x - 15$ (b) $3y = 15 - 5x$
(c) $3y = 5x + 15$ (d) $3x = 15 + 5y$

Q.80 If the inclination of the line l lies between $]0°, 90°[$, then the slope of l is:
(a) Positive (b) Undefined
(c) Negative (d) None of these

Q.81 The points A, B and C are collinear, then slope of $\overline{AB}$ and slope of $\overline{BC}$ are
(a) Equal (b) Opposite in sign
(c) Different (d) None of these

Q.82 $x = 4$ is a line
(a) Parallel to x-axis (b) Parallel to y-axis
(c) $\perp$ to y-axis (d) None of these

Q.83 The line $x = a$ is on the left to the y-axis,
(a) $a > 0$ (b) $a < 0$
(c) $a = 0$ (d) $a \neq 0$

Q.84 if $l_1 : a_1x + b_1y + c_1$ and $l_2 : a_2x + b_2y + c_2$ are any two distinct and coplanar lines, then l_1 and l_2 are perpendicular if:
(a) $a_1a_2 + b_1b_2 = 0$ (b) $a_1b_2 + b_1a_2 = 0$
(c) $a_1b_2 - b_1a_2 = 0$ (d) $a_1a_2 - b_1b_2 = 0$

Q.85 The distance d from the point $P(x_1, y_1)$ to the line $ax + by + c = 0$ is given by
(Lahore Board 2018 G-II)
(a) $d = \frac{|ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$ (b) $\frac{|ax_1 + by_1 + c|}{\sqrt{a + b}}$
(c) $d \equiv \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 - b^2}}$ (d) $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a - b}}$

Q.86 The coordinate axes divide the plane into --------- equal parts.
(a) 1 (b) 2
(c) 3 (d) 4

Q.87 The two lines l_1 and l_2 with respective slopes m_1 and m_2 are perpendicular if and only if:
(Sahiwal Board 2017)(Rawalpindi Board 2017)
(a) $m_1 = m_2$ (b) $m_1 \times m_2 + 1 = 0$
(c) $m_1m_2 = 1$ (d) $m_1 \times m_2 = 0$

Q.88 The line $y = a$ is below the x-axis, if
(a) $a > 0$ (b) $a < 0$
(c) $a \neq 0$ (d) $a = 0$

Q.89 The slope of line $y = \frac{1}{2}x + \frac{1}{3}$ is equal to
(a) $-\frac{1}{3}$ (b) $\frac{1}{2}$
(c) $-\frac{1}{2}$ (d) $\frac{1}{3}$

Q.90 If the line l is parallel to y-axis, then the inclination of l is:
(a) 0° (b) 90°
(c) 180° (d) Undefined

Q.91 Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ will be identical if

(a) $a_1a_2 = b_1b_2 = c_1c_2$

(b) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(c) $a_1a_2 + b_1b_2 + c_1c_2 \neq 0$

(d) $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Q.92 Any line perpendicular to the line $ax + by + c = 0$ is:

(a) $bx + ay + c' = 0$ (b) $cx - ay + a' = 0$

(c) $bx - ay + c' = 0$ (d) None of these

Q.93 The point (x_1, y_1) below the line $ax + by + c = 0$ if

(a) $ax_1 + by_1 + c < 0, b < 0$

(b) $ax_1 + by_1 + c > 0, b > 0$

(c) $ax_1 + by_1 + c = 0, b > 0$

(d) $ax_1 + by_1 + c < 0, b > 0$

Q.94 The equation to the straight line which passes through the point (2, 9) and makes an angle of 45° with x-axis is

(a) $x + y + 7 = 0$ (b) $x - y + 7 = 0$

(c) $y - x + 7 = 0$ (d) None of these

Q.95 The area of the triangle ABC whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ is given by

(a) $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & x \\ x_2 & y_2 & y \\ x_3 & y_3 & z \end{vmatrix}$ (b) $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

(c) $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 2 \end{vmatrix}$ (d) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Q.96 The equation of a line which passes the point (3, 4) and whose intercepts on the axes equal in magnitude but opposite in sign is:

(a) $x + y - 1 = 0$ (b) $x - y - 1 = 0$

(c) $y - x + 1 = 0$ (d) None of these

Q.97 The equation of a straight line which parallel to the line $3x - 2y + 5 = 0$ and passes through $(2, -1)$ is:

(a) $3x + 2y - 8 = 0$ (b) $3x - 2y + 8 = 0$

(c) $3x - 2y - 8 = 0$ (d) $3x + 2y + 8 = 0$

Q.98 The equation of the straight line which passes through the point $(1, -3)$ and makes an intercept on the y-axis twice as long as that on the x-axis is:

(a) $2x + y - 1 = 0$ (b) $2x + y + 1 = 0$

(c) $2x - y - 1 = 0$ (d) None of these

Q.99 Two non-parallel lines intersect each other at.

(Lahore Board 2017 G-I)

(a) 1 point (b) 0 point

(c) ∞ points (d) 2 points

Q.100 The inclination of $x = y$ is.

(Lahore Board 2017 G-II)

(a) 30° (b) 60°

(c) 45° (d) 180°

Q.101 If $a \neq 0$ and $b \neq 0$, then y intercept of the line $ax + by + c = 0$. *(Multan Board 2017 G-I)*

(a) $\frac{b}{c}$ (b) $-\frac{b}{c}$

(c) $-\frac{c}{b}$ (d) $\frac{c}{b}$

Q.102 Inclination of line joining two points (–2, 4) and (5, 11) equals. *(Rawalpindi Board 2017 G-I)*

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

Q.103 Perpendicular distance of P(6, –1) from the line $3x + 4y + 1 = 0$ equals.

(Rawalpindi Board 2017 G-I)

(a) 1 (b) 2

(c) 3 (d) 4

Q.104 $x \cos \alpha + y \sin \alpha = p$ is equation of straight line is. *(Sargodha Board 2017)*

(a) Slope intercept form

(b) Two intercept from

(c) Point slope form (d) Normal

Q.105 Slope of a line which is perpendicular to the line $ax + by + c = 0$ is.

(Multan Board 2018 G-I)

(a) $-\frac{a}{b}$ (b) $\frac{b}{a}$

(c) $-\frac{b}{a}$ (d) $\frac{a}{b}$

Q.106 If λ_1 with slope $\frac{-1}{2}$ and λ_2 with slope 2, then the angle between λ_1 and λ_2 lines is.

(Multan Board 2017 G-I)

(a) 45° (b) 60°

(c) 90° (d) None of these

Q.107 An equation of the horizontal line through P(7, –9). *(A.J.K Board 2017)*

(Lahore Board 2018 G-II)

(a) $y = -9$ (b) $y = 9$

(c) $x = 7$ (d) $x = -7$

Q.108 A linear equation in two variables represents.

(A.J.K Board 2017)

(a) Circle (b) Ellipse

(c) Hyperbola (d) Straight line

Q.109 Slope of line $ax - by + c = 0$ is.
(Bahawalpur Board 2018)
(a) $\frac{a}{b}$ (b) $\frac{b}{a}$
(c) $-\frac{a}{b}$ (d) $-\frac{b}{a}$

Q.110 x-intercept and y-intercept for the line $2x - y + 4 = 0$ are *(Bahawalpur Board 2018)*
(a) (2, –4) (b) (–2, –4)
(c) (–2, 4) (d) (2, 4)

Q.111 The distance of point (2, 3) from y-axis is.
(Faisalabad Board 2018)
(a) 2 (b) 3
(c) 5 (d) $\sqrt{13}$

Q.112 If P(7, –2) lies on circle with center (–5, 3), then its in radius is. *(Gujranwala Board 2018)*
(a) 13 (b) $\sqrt{13}$
(c) 17 (d) $\sqrt{17}$

Q.113 Co-ordinates of mid point of A(–1, 4) , B(6, 2).
(Gujranwala Board 2018)
(a) (–7, 2) (b) (7, –2)
(c) $\left(\frac{5}{2}, 3\right)$ (d) $\left(3, \frac{5}{2}\right)$

Q.114 If a line meets x and y axis at 2, 3, then its equation is. *(Gujranwala Board 2018)*
(a) $2x + 3y = 0$ (b) $3x + 2y = 0$
(c) $\frac{x}{2} + \frac{y}{3} = 0$ (d) $\frac{x}{2} + \frac{y}{3} = 1$

Q.115 If α is the inclination of a line "l" then it must be true that. *(Lahore Board 2018 G-I)*
(a) $0 \leq \alpha < \frac{\pi}{2}$ (b) $\frac{\pi}{2} \leq \alpha < \pi$
(c) $0 < \alpha < \pi$ (d) $0 \leq \alpha < 2\pi$

Q.116 Inclination of a line perpendicular to y-axis is.
(Multan Board 2018 G-I)
(a) 0° (b) 60°
(c) 30° (d) 90°

Q.117 Slope of line $5x + 7y = 35$ is.
(Multan Board 2018 G-II)
(a) $\frac{5}{7}$ (b) $\frac{7}{5}$
(c) 35 (d) $-\frac{5}{7}$

Q.118 A Equation of line with slope –2, y-intercept 3 is
(Multan Board 2018 G-II)
(a) $x - 2y = 3$ (b) $3x + 2y = 2$
(c) $2x + y = 3$ (d) $x + 3y = 2$

Q.119 The line bisecting the 2nd and 4th quadrant.
(Lahore Board 2018 G-II)
(a) $y = x$ (b) $y = -x$
(c) $y = \frac{1}{x}$ (d) $x + y = 1$

Q.120 The line bisecting 1st and 3rd quadrant.
(Rawalpindi Board 2018)
(a) $y = x$ (b) $y = -x$
(c) $y = x + c$ (d) $xy = c$

Q.121 Slope of line which is perpendicular to the line $2x - 4y + 11 = 0$ is. *(Rawalpindi Board 2018)*
(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$
(c) 2 (d) –2

Q.122 Equation of a line passing through (5, –7) having slope undefined is.
(Sahiwal Board 2018)
(a) $y = -7$ (b) $x = 5$
(c) $x = -5$ (d) $y = 7$

Q.123 The line $ax + by + c = 0$ is parallel to x-axis if.
(Sahiwal Board 2018)
(a) $a = 0$ (b) $b = 0$
(c) $c = 0$ (d) $b = c$

Q.124 Distance of point (x, y) from y-axis is.
(Sahiwal Board 2018)
(a) $|x|$ (b) $|y|$
(c) x (d) y

Q.125 The line $ax + by + c = 0$ is parallel to y-axis if.
(Sargodha Board 2018)
(a) $c = 0$ (b) $a = 0$
(c) $a = b$ (d) $b = 0$

Q.126 Equation of line bisecting 1st and 2nd quadrant.
(Rawalpindi Board 2018)
(a) $y = x$ (b) $y = -x$
(c) $y = x + c$ (d) $xy = c$

Q.127 Equation of line bisecting 2nd and 4th quadrant.
(Lahore Board 2018 G-I)
(a) $y = x$ (b) $y = -x$
(c) $y = \frac{1}{x}$ (d) $x + y = 1$

Q.128 The perpendicular distance of the line $3x + 4y + 5 = 0$ from the origin is:
(Bahawalpur Board 2019)
(a) 0 (b) 1
(c) 2 (d) 5

Q.129 Equation of the line having slope – 5 and y - intercept - 7 is: *(Bahawalpur Board 2019)*
(a) $5x + y + 7 = 0$ (b) $5x - y + 7 = 0$
(c) $5x + y - 7 = 0$ (d) $7x + y + 5 = 0$

Q.130 When a line intersects the y - axis at (0, 4) then y - intercept is: *(Bahawalpur Board 2019)*

(a) 4 (b) 2
(c) 0 (d) 6

Q.131 Slope of the Line Perpendicular to line $2x - 3y + 1 = 0$ is equal to: *(Bahawalpur Board 2019)*

(a) $\frac{3}{2}$ (b) $-\frac{3}{2}$
(c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

Q.132 Slope – intercept form of line is: *(Sahiwal Board 2019)*

(a) $y - y_1 = m(x - x_1)$ (b) $\frac{x}{a} + \frac{y}{b} = 1$
(c) $y = mx + c$ (d) $x\cos\alpha + y\sin\alpha = p$

Q.133 The line $2x - y - 4 = 0$ cuts x-axis at point: *(Faisalabad Board 2019)*

(a) (2 , 0) (b) (0 , –2)
(c) (0, –4) (d) (4 , 0)

Q.134 The equation of line $\frac{x}{a} + \frac{y}{b} = 1$ is: *(Faisalabad Board 2019)*

(a) Normal form (b) Intercepts form
(c) Point-slope form (d) Two-points form

Q.135 The perpendicular distance of a line $12x + 5y - 7 = 0$ from origin is: *(Faisalabad Board 2019)*

(a) $\frac{1}{13}$ (b) $\frac{13}{7}$
(c) $\frac{7}{13}$ (d) 13

Q.136 Intercept form of equation of a line is: *(Faisalabad Board 2019 G-II)*

(a) $\frac{x}{a} - \frac{y}{b} = 0$ (b) $\frac{x}{a} + \frac{y}{b} = 0$
(c) $\frac{x}{a} - \frac{y}{b} = 4$ (d) $\frac{x}{a} + \frac{y}{b} = 1$

Q.137 Slope of the line parallel to x-axis is: *(Faisalabad Board 2019 G-II)*

(a) Undefined (b) 1
(c) 0 (d) –1

Q.138 If A(–3, 6) and B(3, 2), then slope of AB is *(Gujranwala Board 2019 G-I)*

(a) $\frac{3}{2}$ (b) $\frac{-2}{3}$
(c) $\frac{1}{3}$ (d) $\frac{-3}{2}$

Q.139 Slope of the line $2x + y - 3 = 0$ is *(Gujranwala Board 2019 G-II)*

(a) 2 (b) $\frac{2}{3}$
(c) –2 (d) $-\frac{2}{3}$

Q.140 The perpendicular distance of a line $5x + 12y = 7$ from origin is: *(Lahore Board 2019)*

(a) $\frac{1}{13}$ (b) $\frac{13}{7}$
(c) $\frac{7}{13}$ (d) –7

Q.141 y-intercept of the line $2x - y - 4 = 0$ is: *(Lahore Board 2019)*

(a) 2 (b) –2
(c) 4 (d) –4

Q.142 The equation of line $\frac{x}{b} + \frac{y}{a} = 1$ is in: *(Lahore Board 2019)*

(a) Normal form (b) Intercept form
(c) Point-slope form (d) Two-points form

Q.143 Distance of line $x + 2y + 5 = 0$ from origin is: *(Multan Board 2019)*

(a) 1 (b) $\sqrt{5}$
(c) 5 (d) 2

Q.144 Length of perpendicular from (1 , 1) to the line $4x - 3y + 9 = 0$ equals: *(Multan Board 2019)*

(a) 2 (b) 4
(c) 3 (d) 9

Q.145 Equation of horizontal line through (2, 3) is: *(Multan Board 2019)*

(a) $y = 3$ (b) $y = 2$
(c) $x \cong 3$ (d) $x = 2$

Q.146 Slope of vertical line is: *(Multan Board 2019)*

(a) 0 (b) 1
(c) ∞ (d) 2

Q.147 Slope of line AB, A(1 , 2), B(1 , 4) *(Sahiwal Board 2019)*

(a) 0 (b) 1
(c) 2 (d) undefined

Q.148 If a line "λ" is parallel to x - axis, then inclination = *(Rawalpindi Board 2019)*

(a) 90° (b) 0°
(c) 30° (d) 45°

Q.149 If a line"λ" intersect x - axis at(α, 0), then "α" is called _____ of line "λ".

(Rawalpindi Board 2019)

(a) y-intercept (b) x – intercept

(c) slope (d) inclination

Q.150 y = mx + c is _____ form of equation of line: *(Rawalpindi Board 2019)*

(a) point slope (b) intercept

(c) normal (d) slope intercept

Q.151 An equation of line bisecting I and III quadrant is: *(Rawalpindi Board 2019)*

(a) x = y (b) x = – y

(c) x + 2y = 0 (d) x – 2y = 0

Q.152 The angle form the line with slope 2 to the line with slope 1 will be *(Sargodha Board 2019)*

(a) $\tan^{-1}(3)$ (b) $\tan^{-1}\left(\frac{1}{3}\right)$

(c) $\text{Cos}^{-1}(3)$ (d) $\text{Cos}^{-1}\left(\frac{1}{3}\right)$

Q.153 Equation of straight line passing through (0 , 0) and parallel to the line with slope 2 will be *(Sargodha Board 2019)*

(a) $x = \frac{2}{3}y$ (b) x = y

(c) $y = \frac{1}{2}x$ (d) y = 2x

Q.154 Equation of vertical line is *(Sahiwal Board 2019)*

(a) y = c (b) y = –c

(c) x = c (d) y = x

Q.155 Distance of A (1, 1) from origin is *(Sahiwal Board 2019)*

(a) 2 (b) $\sqrt{2}$

(c) 0 (d) 1

Q.156 The perpendicular distance of the line 12x + 5y = 7 from the origin is *(Gujranwala Board 2019 G-II)*

(a) $\frac{7}{13}$ (b) $\frac{13}{7}$

(c) 13 (d) $\frac{1}{13}$

EXERCISE 4.4

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find p such that the lines 2x – 3y – 1 = 0, 3x – y – 5 = 0 and 3x + py + 8 = 0 are concurrent.
(Gujranwala Board 2013)(Multan Board 2014 G-I)
(Bahawalpur Board 2018)(Multan Board 2018 G-I)

Ans. As given lines are concurrent so.

$$\begin{vmatrix} 2 & -3 & -1 \\ 3 & -1 & -5 \\ 3 & P & 8 \end{vmatrix} = 0$$

$2(-8 + 5P) + 3(24 + 15) - 1(3P + 3) = 0$

$2(-8 + 5P) + 3(39) - 1(3P + 3) = 0$

$-16 + 10P + 117 - 3P - 3 = 0$

$7P + 98 = 0 \Rightarrow P = \frac{-98}{7}$

$\boxed{P = -14}$

Q.2 Find the condition that the lines $y = m_1x + c_1, y = m_2x + c_2$, $y = m_3x + c_3$ are concurrent.
(Lahore Board 2011 G-II, 2015 G-II)

Ans. Given $m_1x - y + c_1 = 0$ .....(1)

$m_2x - y + c_2 = 0$ .....(2)

$m_3x - y + c_3 = 0$ .....(3)

Lines (1) , (2) , (3) will be concurrent if

$$\begin{vmatrix} m_1 & c_1 & 1 \\ m_2 & c_2 & 1 \\ m_3 & c_3 & 1 \end{vmatrix} = 0$$

Applying $R_2 - R_1$ and $R_3 - R_1$, we have

$$\begin{vmatrix} m_1 & c_1 & 1 \\ m_2 - m_1 & c_2 - c_1 & 1 - 1 \\ m_3 - m_1 & c_3 - c_1 & 1 - 1 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} m_1 & c_1 & 1 \\ m_2 - m_1 & c_2 - c_1 & 0 \\ m_3 - m_1 & c_3 - c_1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} m_2 - m_1 & c_2 - c_1 \\ m_3 - m_1 & c_3 - c_1 \end{vmatrix} = 0$$

[by expandng from column 3]

$\Rightarrow (m_2 - m_1)(c_3 - c_1) - (m_3 - m_1)(c_2 - c_1) = 0$

$\Rightarrow (m_2 - m_1)(c_3 - c_1) = (m_3 - m_1)(c_2 - c_1)$

is the required condition.

LONG QUESTIONS

Q.1 Determine the value of p, such that the lines

$2x - 3y - 1 = 0$, $3x - y - 5 = 0$ and $3x + py + 8 = 0$

(Multan Board 2014 G-II)

Sol: See Short Question 6

Q.2 Find an equation of the line through the intersection of the lines $x + 2y + 3 = 0$,

$3x + 4y + 7 = 0$ and making equal intercepts on axis.

(Gujranwala Board 2014)(Multan Board 2017 G-I)

(D.G.K Board 2017 G-I,II)

Sol: $x + 2y + 3 = 0$ ..... (1)

$3x + 4y + 7 = 0$ ..... (2)

3 Eq (1) – Eq (2)

$3x + 6y + 9 = 0$

$\pm 3x \pm 4y \pm 7 = 0$

$2y + 2 = 0$

$y = -1$

(1) $\Rightarrow x - 2 + 3 = 0 \Rightarrow x = -1$

$(-1, -1)$ is the pint of intersection of (1) and (2)

Let equation of required line is

$\frac{x}{a} + \frac{y}{b} = 1$

By given condition $a = b$

$\therefore \frac{x}{a} + \frac{y}{a} = 1 \Rightarrow x + y = a$ ..... (3)

as it passes through $(-1, 1)$

$\therefore \quad -1 - 1 = a \Rightarrow a = -2$

(3) $\Rightarrow x + y = -2$

$\Rightarrow x + y + 2 = 0$

Which is required equation of line.

Q.3 Find the angle of the triangle whose vertices are A(–5, 4), B(–2, –1), C(7, –5).

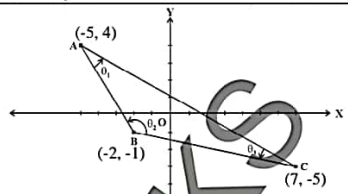
(Bahawalpur Board 2016)(Lahore Board 2018 G-I)

Sol: Given that: A(–5, 4), B(–2, –1), C(7, –5)

Slope of AB $= m_1 = \frac{-1-4}{-2-(-5)} = \frac{-5}{-2+5} = \frac{-5}{3}$

Slope of BC $= m_2 = \frac{-5-(-1)}{7-(-2)} = \frac{-5+1}{7+2} = \frac{-4}{9}$

Slope of AC $= m_3 = \frac{-5-4}{7-(-5)} = \frac{-9}{7+5} = \frac{-9}{12} = \frac{-3}{4}$



Now,

$\tan\theta_1 = \frac{m_3 - m_1}{1 + m_3 \cdot m_1} \Rightarrow \tan\theta_1 = \frac{\frac{-3}{4} - \frac{-5}{3}}{1 + \left(\frac{-3}{4}\right)\left(\frac{-5}{3}\right)}$

$\tan\theta_1 = \frac{\frac{-3}{4} + \frac{5}{3}}{1 + \frac{15}{12}} \Rightarrow \tan\theta_1 = \frac{\frac{-9+20}{12}}{\frac{12+15}{12}}$

$\tan\theta_1 = \frac{\frac{11}{12}}{\frac{27}{12}} \Rightarrow \tan\theta_1 = \frac{11}{27}$

$\theta_1 = \tan^{-1}\left(\frac{11}{27}\right) \Rightarrow \theta_1 = 22.17°$

$\tan\theta_2 = \frac{m_1 - m_2}{1 + m_1 \cdot m_2} \Rightarrow \tan\theta_2 = \frac{\frac{-5}{3} - \frac{-4}{9}}{1 + \left(\frac{-5}{3}\right)\left(\frac{-4}{9}\right)}$

$\tan\theta_2 = \frac{-\frac{5}{3} + \frac{4}{9}}{1 + \frac{20}{27}} \Rightarrow \tan\theta_2 = \frac{\frac{-45+12}{27}}{\frac{27+20}{27}}$

$\tan\theta_2 = \frac{\frac{-33}{27}}{\frac{47}{27}} \Rightarrow \tan\theta_2 = -\frac{33}{47}$

$\theta_2 = \tan_{-1}\left(-\frac{33}{47}\right) \Rightarrow \theta_2 = 144.93°$

and $\theta_1 + \theta_2 + \theta_3 = 180°$

$\Rightarrow 22.17° + 144.93° + \theta_3 = 180°$

$167.1° + \theta_3 = 180°$

$\Rightarrow \theta_3 = 180° - 167.1° \Rightarrow \theta_3 = 12.9°$

Q.4 Find the interior angles of a triangle whose vertices are A(6, 1), B(2, 7) and C(–6, –7).

(Faisalabad Board 2016)

Sol: A(6, 1) , B(2, 7), C(–6, –7)

Slope of AB $= m_1 = \frac{7-1}{2-6} = \frac{6}{-4} = \frac{-3}{2}$

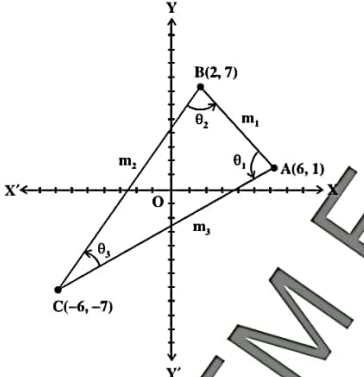
Slope of BC $= m_2 = \frac{-7-7}{-6-2} = \frac{-14}{-8} = \frac{7}{4}$

Slope of AC $= m_3 = \frac{-7-1}{-6-6} = \frac{-8}{-12} = \frac{2}{3}$

$$\tan\theta_1 = \frac{m_3 - m_1}{1 + m_3 . m_1} = \frac{\frac{2}{3} + \frac{3}{2}}{1 + \left(\frac{2}{3}\right)\left(\frac{-3}{2}\right)}$$

$$= \frac{\frac{2}{3} + \frac{3}{2}}{1 - \frac{6}{6}} = \frac{\frac{4+9}{6}}{\frac{6-6}{6}} = \frac{13}{0}$$

$\theta_1 = \tan^{-1}\infty \Rightarrow$ **$\theta_1 = 90°$**



$$\tan\theta_2 = \frac{m_1 - m_2}{1 + m_1 . m_2} = \frac{-\frac{3}{2} - \frac{7}{4}}{1 + \left(-\frac{3}{2}\right)\left(\frac{7}{4}\right)}$$

$$= \frac{-\frac{3}{2} - \frac{7}{4}}{1 - \frac{21}{8}} = \frac{\frac{-6-7}{4}}{\frac{8-21}{8}} = \frac{\frac{-13}{4}}{\frac{-13}{8}}$$

$$= \frac{-13}{4} \times \frac{8}{-13}$$

$\theta_2 = \tan^{-1}(2) \Rightarrow$ **$\theta_2 = 63.43°$**

As we know

$\theta_3 = 180° - \theta_1 - \theta_2 \Rightarrow \theta_3 = 180° - 90° - 63.43°$

$\theta_3 = 180° - 153.43° \Rightarrow$ **$\theta_3 = 26.57°$**

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 The point of intersection concurrency of median of a triangle is:
(Gujranwala Board 2013)(Sargodha Board 2016) (Lahore Board 2015 G-I)(Lahore Board 2014 G-II) (Rawalpindi, Faisalabad Board 2018)
(a) Centroid (b) Orthocentre
(c) Circumference (d) Incentre

Q.2 The point of intersection of the altitudes of a triangle is called: *(Multan Board 2018 G-I) (D.G.K Board 2012 G-I, Sahiwal Board 2013)*
(a) Circumcentre (b) in-centre
(c) Centroid (d) Orthocentre

Q.3 The point of intersection of angle bisectors of a triangle is:
(D.G.K Board 2010, Multan Board 2013 G-II) (Lahore Board 2012 G-II)(D.G.K Board 2011)
(a) In-centre (b) Centroid
(c) Circumcentre (d) Orthocentre

Q.4 The point of intersection of two lines $x + y = 2$ and $2x - y = 1$ is: *(Multan Board 2014 G-I) (D.G.K Board 2011)*
(a) (1, 2) (b) (−1, 2)
(c) (−1, −2) (d) (1, 1)

Q.5 The centroid of a triangle divides each median in the ratio: *(Lahore Board 2018 G-I) (D.G.K Board 2014 G-II, D.G.K Board 2015 G-I) (Sargodha Board 2017)(Rawalpindi Board 2016)*
(a) 2:1 (b) 1:2
(c) 3:1 (d) 1:3

Q.6 Number of straight lines passing through one point is: *(Gujranwala Board 2014)*
(a) One (b) Finite
(c) Infinite (d) Three

Q.7 X-co-ordinate of centroid of triangle ABC with A(−2, 3), B(−4, 1), C(3, 5) equals. *(Lahore Board 2014 G-I)*
(a) −1 (b) 1
(c) 3 (d) −3

Q.8 Y-co-ordinate of centroid of triangle with vertices A(−2, 3) , B(−4, 1) and C(3, 5) is. *(Sahiwal Board 2018)*
(a) 9 (b) 3
(c) $\frac{9}{2}$ (d) $\frac{3}{2}$

Q.9 An infinite number of lines can pass through.
(a) One point (b) Three point
(c) Two point (d) None of these

Q.10 The centroid of a triangle is a point that divides each median in the ratio.

(a) 2 : 1 (b) 2 : 3

(c) 1 : 3 (d) 4 : 3

Q.11 Then co-ordinates of its centroid are given by

(a) $(x_1 + x_2 + x_3, y_1 + y_2 + y_3)$

(b) $\left(\frac{x_1 - x_2 - x_3}{3}, \frac{y_1 - y_2 - y_3}{3}\right)$

(c) $\left(\frac{x_1 + x_2 + x_3}{2}, \frac{y_1 + y_2 + y_1}{2}\right)$

(d) $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_1}{3}\right)$

Q.12 Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ will intersect if

(a) $a_1b_2 - a_2b_1 = 0$ (b) $a_1b_2 - a_2b_1 \neq 0$

(c) $a_1a_2 - b_1b_2 = 0$ (d) $a_1a_2 - b_1b_2 \neq 0$

Q.13 The medians of a triangle are. *(Faisalabad Board 2017)*

(a) Collinear (b) Concurrent

(c) Perpendicular (d) Parallel

Q.14 The point of concurrency of altitudes of a triangle is called. *(Gujranwala Board 2019 G-II)*

(a) centroid (b) orthocentre

(c) in centre (d) circum centre

EXERCISE 4.5

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find the equations of lines represented by $3x^2 + 7xy + 2y^2 = 0$

(D.G Khan Board 2014 G-I)(Faisalabad Board 2017)

Ans. $3x^2 + 7xy + 2y^2 = 0$ ..... (1)

Divide both sides by x^2

$$\frac{3x^2}{x^2} + \frac{7xy}{x^2} + \frac{2y^2}{x^2} = 0$$

or $$3 + 7\frac{y}{x} + 2\frac{y^2}{x^2} = 0$$

or $$2\left(\frac{y}{x}\right)^2 + 7\left(\frac{y}{x}\right) + 3 = 0$$

$$\frac{y}{x} = \frac{-7 \pm \sqrt{(7)^2 - 4(2)(3)}}{2(2)}$$

[By quadratic formula]

$$= \frac{-7 \pm \sqrt{49 - 24}}{4} = \frac{-7 \pm \sqrt{25}}{4} = \frac{-7 \pm 5}{4}$$

$$= -\frac{12}{4}, -\frac{2}{4} = -3, -\frac{1}{2}$$

The two lines represented by (1) have equations

$y = -3x$ and $y = -\frac{1}{2}x$ or $3x + y = 0$ and $x + 2y = 0$

Q.2 Find the equations of lines represented by $20x^2 + 17xy - 24y^2 = 0$

(Lahore Board 2011 G-I)(Gujranwala Board 2012) (Sargodha Board 2016)(Multan Board 2018 G-II) (Rawalpindi Board 2019)

Ans. Dividing given equation by x^2, we have

$$\frac{20x^2}{x^2} + \frac{17xy}{x^2} - \frac{24y^2}{x^2} = 0$$

$$\Rightarrow \quad 20 + 17\left(\frac{y}{x}\right) - 24\left(\frac{y}{x}\right)^2 = 0$$

Multiplying both sides by -1, we get

$$24\left(\frac{y}{x}\right)^2 - 17\left(\frac{y}{x}\right) - 20 = 0$$

It is quadratic in $\left(\frac{y}{x}\right)$, using quadratic formula, we have

$$\frac{y}{x} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-17) \pm \sqrt{(-17)^2 - 4(24)(-20)}}{2(24)}$$

$$= \frac{17 \pm \sqrt{289 + 1920}}{48}$$

$$= \frac{17 \pm 47}{48} = \frac{17 + 47}{48}, \frac{17 - 47}{48}$$

$$= \frac{64}{48}, \frac{-30}{48} = \frac{4}{3}, \frac{-5}{8}$$

$$\Rightarrow \quad \frac{y}{x} = \frac{4}{3}, \quad \frac{y}{x} = -\frac{5}{8}$$

$$\Rightarrow \quad y = \frac{4}{3}x \quad \text{and} \quad y = -\frac{5}{8}x$$

$$\Rightarrow \quad 3y = 4x \text{ and } 8y = -5x$$

$$\Rightarrow \quad 4x - 3y = 0 \text{ and } 5x + 8y = 0.$$

Q.3 Find the angle between the lines represented by $2x^2 + 3xy - 5y^2 = 0$

(Lahore Board 2010 G-II, 2011 G-II) (Gujranwala Board 2016)

Ans. $2x^2 + 3xy - 5y^2 = 0$ ..... (1)

To find measure of the angle between the lines represented by (1), compare (1) with $ax^2 + 2hxy + by^2 = 0$, we get

$$a = 2, h = \frac{3}{2}, b = -5$$

If θ is the measure of the angle between the given lines, then

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b} = \frac{2\sqrt{\left(\frac{3}{2}\right)^2-(2)(-5)}}{2-5}$$

$$= \frac{2\sqrt{\frac{9}{4}+10}}{-3} = \frac{2\sqrt{\frac{9+40}{4}}}{-3} = \frac{2\sqrt{\frac{49}{4}}}{-3}$$

$$= \frac{2\times\frac{7}{2}}{-3} = -\frac{7}{3} \Rightarrow \theta = \tan^{-1}\left(-\frac{7}{3}\right) = 113.2°$$

Q.4 Find measure of angle between the lines represented by $x^2 - xy - 6y^2 = 0$.
(Gujranwala Board 2016)(Rawalpindi Board 2016) (Lahore Board 2016 G-I)(Faisalabad Board 2016) (Sahiwal Board 2013, 2019)

Ans. Here $a = 1, 2h = -1$ or $h = -\frac{1}{2}, b = -6$

If θ is measure of the angle between the given lines, then

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b} = \frac{2\sqrt{\frac{1}{4}+6}}{-5} = -1$$

$\Rightarrow \theta = 135°$

Acute angle between the lines $= 180° - \theta$
$= 180° - 135° = 45°$

LONG QUESTIONS

Q.1 Find lines represented by $2x^2 + 3xy - 5y^2 = 0$, also find measure of angle between them.
(Sahiwal Board 2014)

Sol: See Short Question 7

Q.2 Find a joint equation of the straight line through the origin perpendicular to the lines represented by $x^2 + xy - 6y^2 = 0$. *(Lahore Board 2015, 2018 G-II)*

Sol: Let $y = m_1x$ and $y = m_2x$ be the equations of the lines represented by $x^2 + xy - 6y^2 = 0$ then

$$m_1 + m_2 = \frac{-2h}{b} = \frac{-1}{-6} = \frac{1}{6}$$

and $m_1m_2 = \frac{a}{b} = \frac{1}{-6} = \frac{-1}{6}$

The equation of the line passing through the origin and perpendicular to $y = m_1x$ is

$$y - 0 = \frac{-1}{m_1}(x - 0)$$

$$y = \frac{-1}{m_1}x \Rightarrow m_1y = -x$$

$\Rightarrow x + m_1y = 0$ ..... (1)

The equation of the line passing through the origin and perpendicular to $y = m_2x$ is

$$y - 0 = \frac{-1}{m_2}(x - 0)$$

$$y = \frac{-1}{m_2}x \Rightarrow m_2y = -x$$

$\Rightarrow x + m_2y = 0$ ..... (2)

The joint equation of the lines through (1) and (2) is

$$(x + m_1y)(x + m_2y) = 0$$
$$x^2 + xym_1 + xym_2 + m_1m_2y^2 = 0$$
$$x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$$
$$x^2 + \frac{1}{6}xy - \frac{1}{6}y^2 = 0$$
$$6x^2 + xy - y^2 = 0$$

Q.3 Find a joint equation of the lines through the origin and perpendicular to the lines represented by $x^2 - 2xy\tan\alpha - y^2 = 0$.
(Sahiwal Board 2018)(Bahawalpur Board 2019)

Sol: Let $y = m_1x$ and $y = m_2x$ be the equations of the lines represented by $x^2 - 2xy\tan\alpha - y^2 = 0$ then

$$m_1 + m_2 = \frac{-2h}{b} = \frac{2\tan\alpha}{-1} = -2\tan\alpha$$

and $m_1m_2 = \frac{a}{b} = \frac{1}{-1} = -1$

The equation of the line passing through the origin and perpendicular to $y = m_1x$ is

$$y - 0 = \frac{-1}{m_1}(x - 0)$$

$$y = \frac{-1}{m_1}x \Rightarrow m_1y = -x$$

$\Rightarrow x + m_1y = 0$ ..... (1)

The equation of the line passing through the origin and perpendicular to $y = m_2x$ is

$$y - 0 = \frac{-1}{m_2}(x - 0)$$

$$y = \frac{-1}{m_2}x \Rightarrow m_2y = -x$$

$\Rightarrow x + m_2y = 0$ ..... (2)

The joint equation of the lines through (1) and (2) is

$$(x + m_1y)(x + m_2y) = 0$$
$$x^2 + xym_1 + xym_2 + m_1m_2y^2 = 0$$
$$x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$$
$$x^2 + (-2\tan\alpha)xy - 1y^2 = 0$$
$$x^2 - 2\tan\alpha xy - y^2 = 0$$

Q.4 Find a joint equation of the lines through the origin and perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$. *(Multan Board 2018 G-I)*

Sol: Let $y = m_1x$ and $y = m_2x$ be the equations of the lines represented by $ax^2 + 2hxy + by^2 = 0$ then

$$m_1 + m_2 = \frac{-2h}{b} \quad \text{and} \quad m_1m_2 = \frac{a}{b}$$

The equation of the line passing through the origin and perpendicular to $y = m_1x$ is

$$y - 0 = \frac{-1}{m_1}(x - 0)$$

$$y = \frac{-1}{m_1}x \Rightarrow m_1y = -x$$

$$\Rightarrow x + m_1y = 0 \quad \ldots\ldots\ldots (1)$$

The equation of the line passing through the origin and perpendicular to $y = m_2x$ is

$$y - 0 = \frac{-1}{m_2}(x - 0)$$

$$y = \frac{-1}{m_2}x \Rightarrow m_2y = -x$$

$$\Rightarrow x + m_2y = 0 \quad \ldots\ldots\ldots (2)$$

The joint equation of the lines through (1) and (2) is

$$(x + m_1y)(x + m_2y) = 0$$
$$x^2 + xym_1 + xym_2 + m_1m_2y^2 = 0$$
$$x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$$
$$x^2 + \left(\frac{-2h}{b}\right)xy + \frac{a}{b}y^2 = 0$$

Multiplying by b

$$bx^2 - 2hxy + ay^2 = 0$$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 Every homogeneous equation $ax^2 + 2hxy + by^2 = 0$ represents two real line passes through (0, 0) if: *(Multan Board 2012 G-II)*

(a) $h^2 - ab < 0$ (b) $h^2 - ab > 0$
(c) $h^2 > ab$ (d) Both b and c

Q.2 If $f(kx, ky) = k^nf(x, y)$, then $f(x, y) = 0$ is a homogeneous equation of degree. *(Gujranwala Board 2012)(Lahore Board 2013 G-I)*

(a) n + 1 (b) n – 1
(c) n (d) k

Q.3 The lines represented by $ax^2 + 2hxy + by^2 = 0$ are orthogonal if: *(Gujranwala Board 2016) (Faisalabad Board 2018)*

(a) a – b = 0 (b) a + b = 0
(c) a + b > 0 (d) a + b ≠ 0

Q.4 The pair of lines of homogeneous second-degree equation $ax^2 + 2hxy + by^2 = 0$ are real and coincident, if *(Rawalpindi Board 2017 G-II, 2018)*

(a) $h^2 < ab$ (b) $h^2 > ab$
(c) $h^2 = ab$ (d) None of these

Q.5 Two lines of homogeneous second degree equation $ax^2 + 2hxy + by^2 = 0$ are parallel if

(a) $h^2 = ab$ (b) $h^2 > ab$
(c) $h^2 < ab$ (d) None of these

Q.6 A pair of lines of homogeneous second degree equation $ax^2 + 2hxy + by^2 = 0$ are orthogonal, if the sum of the coefficients of x^2 and y^2 is:

(a) 0 (b) 1
(c) 2 (d) 3

Q.7 The slope of a non-vertical line with a as its inclination is defined by

(a) sin α (b) cos α
(c) tan α (d) cot α

Q.8 If the inclination of a line lies between]90°, 180°[, then the slope of line is:

(a) Positive (b) Negative
(c) Zero (d) Undefined

Q.9 The pair of lines of homogeneous second-degree equation $ax^2 + 2hxy + by^2 = 0$ are real and distinct, if *(D.G.K Board 2017 G-I)*

(a) $h^2 < ab$ (b) $h^2 > ab$
(c) $h^2 = ab$ (d) None of these

Q.10 If q is measure of the angle between the pair of line of homogeneous second degree equation $ax^2 + 2hxy + by^2 = 0$ is equal to

(a) $\tan\theta = \frac{2\sqrt{h^2 + ab}}{a + b}$ (b) $\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$

(c) $\tan\theta = \frac{2\sqrt{h^2 - ab}}{a - b}$ (d) $\tan\theta = \frac{2\sqrt{h^2 + ab}}{a + b}$

Q.11 Two lines represented by $ax^2 + 2hxy + by^2 = 0$ are parallel if. *(Lahore Board 2017 G-I)*

(a) $h^2 - ab = 0$ (b) $h^2 + ab = 0$
(c) a + b = 0 (d) a – b

Q.12 Two lines represented by $ax^2 + 2hxy + by^2 = 0$ will be perpendicular if. *(Sargodha 2017) (Rawalpindi Board 2017 G-I)*

(a) $h^2 + ab$ (b) $h^2 - ab$
(c) a – b = 0 (d) a + b ≅ 0

Q.13 The pair of lines of homogeneous second-degree equation $ax^2 + 2hxy + by^2$ are imaginary, if

(a) $h^2 = ab$ (b) $h^2 > ab$
(c) $h^2 < ab$ (d) None of these

Q.14 The lines through origin represented by $ax^2 + 2hxy + by^2 = 0$ are coincident if:

(Faisalabad Board 2019 G-II)

(a) $h^2 = ab$ (b) $h^2 + ab = 0$
(c) $h^2 - ab > 0$ (d) $h^2 - ab < 0$

Q.15 If the straight lines represented by $ax^2 + 2h\,xy + by^2 = 0$ are perpendicular, then

(Sargodha Board 2019)

(a) $h^2 - ab = 0$ (b) $h^2 + ab = 0$
(c) a + b = 0 (d) a – b = 0

Chapter 5

LINEAR INEQUALITIES & LINEAR PROGRAMMING

EXERCISE 5.1

SHORT ANSWERS TO THE QUESTIONS

Q.1 Graph the solution of linear inequality
$2x + y \leq 6$ *(Faisalabad Board 2013, 2019 G-I)*
(Lahore Board 2019 G-II)(Multan Board 2013 G-II)

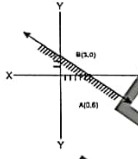
Ans. Give inequality is $2x + y \leq 6$

Associated Eq.

$2x + y = 6$ ..... (i)

Put $x = 0$ in Eq. (i) $\Rightarrow$ $y = 6$ and point A(0, 6) is obtained.

Put $y = 0$ in Eq. (i) $\Rightarrow$ $2x = 6$ and point B(3, 0) is obtained by joining A(0, 6) and B(3, 0) we get a line.



Put 0(0, 0) as test point in $2x + y \leq 6$, we get

$0 \leq 6$, true

So, Solution is all point below the line as shown.

Q.2 Graph the solution set of the following linear inequality in xy-plane: $3x - 2y \geq 6$
(Lahore Board 2013 G-I, II)(Faisalabad Board 2017)
(Gujranwala Board 2013, 2014, 2018, 2019)

Ans. Given inequality is $3x - 2y \geq 6$

Associated Eq. (i) is $3x - 2y = 6$ ..... (i)

For x-intercept

Put, $y = 0$ in Eq. (i) $\Rightarrow$ $3x = 6$

or $x = 2$ and point P(2,0) is obtained

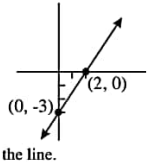
For y-intercept.

Put, $x = 0$ in Eq. (i) $\Rightarrow$ $-2y = 6$ or $y = -3$ and 2nd point is Q(0, –3).

Join P(2, 0) and Q(0, –3) put (0, 0) in $3x - 2y \geq 6$, we get $0 \geq 6$.

It is false.

So we shade opposite to origin side i.e., below the line.



Q.3 Draw the graph of $3x + 2y \geq 6$.
(Gujranwala Board 2013)

Ans. Given equation: $3x + 2y \geq 6$

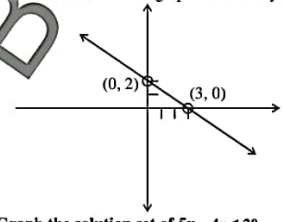
The A.E of (i) is $3x + 2y = 6$ ..... (i)

Put, $x = 0$ in Eq. (i) $\Rightarrow$ $2y = 6$ as $y = 3$ and point is P(0,3)

Put, $y = 0 \Rightarrow 3x = 6$ as $x = 2$ and 2nd point Q(2,0) is obtained.

Join P(0, 3) and Q(2, 0). Put (0, 0) in $3x + 2y \geq 6$ we get, $0 \geq 6$ False

So, we shade above line. The graph is shown by fig.



Q.4 Graph the solution set of $5x - 4y \leq 20$.
(Sargodha Board 2013, 2018, 2019)
(Sahiwal Board 2013)(Multan Board 2016 G-I)
(Gujranwala Board 2010)

Ans. $5x - 4y \leq 20$

The associated Eq. (i) is

$5x - 4y = 20$ ..... (i)

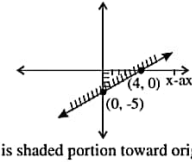
Put $x = 0$ in Eq. (i) $\Rightarrow$ $-4y = 20$ or $y = -5$ and point P(0,-5) is obtained.

Put $= 0$ in Eq. (i) $\Rightarrow$ $5x = 20$ or $x = 4$

And 2nd point Q(4, 0) is obtained Now put O(0,0) as test point in $5x - 4y \leq 20 \Rightarrow$

$0 - 0 \leq 20$

$0 \leq 20$ true



So, solution is shaded portion toward origin as shown.

Q.5 Graph the system of inequalities:

$$x - 2y \le 6$$
$$2x + y \ge 2$$ ***(Lahore Board 2012 G-I)***

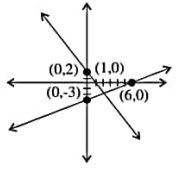
Ans. The corresponding equations are

$x - 2y = 6$ _____ (i), $2x + y = 2$ _____ (ii)

Put $x = 0$ in equation (i), $-2y = 6 \Rightarrow y = -3$

Put $y = 0$ in equation (i), $x = 6$

We draw a straight line (i) passing through $(0, -3)$ and $(6, 0)$ put $(0, 0)$ in equality $x - 2y \le 6$, we get $0 \le 6$, it is true so we shade towards origin, above the straight line (i).



Put $x = 0$ in equation (ii) $y = 2$

Put $y = 0$ in equation (ii) $2x = 2$, $x = 1$

We draw a straight line (ii) passing through (0,2) and (1,0) put $(0, 0)$ in $2x + y \ge 2$, we get $0 \ge 2$, it is false. So we shade away from origin, above the straight line (ii).

Q.6 Graph the feasible region of the system of linear inequalities and find the corner points.

$$x + y \le 5$$
$$-2x + y \ge 2$$
$$x \ge 0$$ ***(Lahore Board 2015 G-II)***

Ans. The corresponding equations are

$x + y = 5$ _____ (i)

$-2x + y = 2$ _____ (ii)

Put $x = 0$ in equation (i), we get $y = 5$

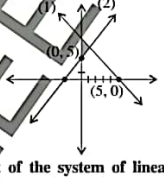
Put $y = 0$ in equation (i), we get $x = 5$

We draw a straight line (i) passing through (0, 5) and (5, 0) Put $(0, 0)$ in $x + y \le 5$, we get $0 \le 5$, it is true so, we shade towards origin i.e., below line (i)

Put $x = 0$ in equation (ii), we get $y = 2$

Put $y = 0$ in equation (ii), we get $x = -1$

We draw a straight line passing through (0, 2) and $(-1, 0)$ put $(0, 0)$ in $-2x + y \ge 2$, we get $0 \ge 2$, it is false. So we shade away from origin (above the line (ii)).



For $x \ge 0$

we shade in the first quadrant.

Q.7 Indicate the solution set of the system of linear inequalities $x + y \ge 5$ and $-y + x \le 1$. ***(Lahore Board 2017 G-I)***

Ans.

The corresponding equation are

$x + y = 5$ ..... (i)

$-y + x = 1$ ..... (ii)

Put $x = 0 \Rightarrow y = 5$

$y = 0 \Rightarrow x = 5$

Join (0, 5) and (5, 0) to draw a line (i).

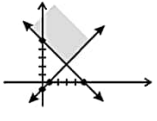
Put (0, 0) in $x + y \ge 5$, we get $0 \ge 5$, it is false. So, we shade above line (i) away from origin.

For equation (2), put $y = 0 \Rightarrow x = 1$

Put $x = 0 \Rightarrow -y = 1 \Rightarrow y = -1$

Joining (1, 0) and (0, –1) to draw a straight line (ii).

Put (0, 0) in $-y + x \le 1$, we get $0 \le 1$, it is true so we shade above line (ii).



Q.8 Graph the linear inequality $2x + 3 \ge 0$

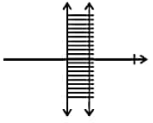
(Faisalabad Board 2019)

Ans. The corresponding equation

$2x + 3 = 0$ or $x = \frac{-3}{2}$

Draw vertical line $x = \frac{-3}{2}$

$x = \frac{-3}{2}$



Put $(0, 0)$ in $2x + 3 \ge 0$

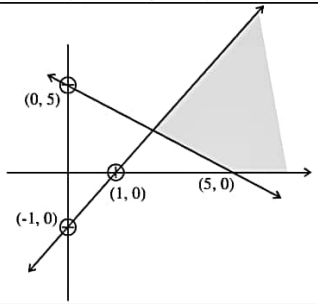
We get $3 \ge 0$, so shade toward right of line.

LONG QUESTIONS

Q.1 Find the solution set of inequalities $x + y \ge 5$ and $x - y \ge 1$, $y \ge 0$ ***(Sahiwal Board 2014)***

Sol: Given that: $x + y \ge 5$, $x - y \ge 1$, $y \ge 0$

$x + y \ge 5$	$x - y \ge 1$	$y \ge 0$
Associated Equation	Associated Equation	Associated Equation
$x + y = 5$ ... (1)	$x - y = 1$ ... (2)	$y = 0$
Put $x = 0$, in eq (1)	Put	Solution region lies on and upper side of x-axis
$0 + y = 5$	$x = 0$, in eq (2)	
$y = 5$	$0 - y = 1$	
$\therefore$ (0, 5)	$y = -1$	
Put $y = 0$, in eq (1)	$\therefore$ (0, –1)	
$x + 0 = 5$	Put	
$x = 5$	$y = 0$, in eq (2)	
$\therefore$ (5, 0)	$x - 0 = 1$	
Test point	$x = 1$	
Put $x = 0$, $y = 0$ in	$\therefore$ (1, 0)	
$x + y > 5$	Test point	
$0 + 0 > 5$	Put	
$0 > 5$ (False)	$x = 0$, $y = 0$ in	
Solution region does not lie on the origin side	$x - y > 1$	
	$0 - 0 > 1$	
	$0 > 1$ (False)	
	Solution region does not lie on the origin side	



MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 An expression involving any one of the symbols <, ≤, >, ≥ is called: *(Faisalabad Board 2013)*
(a) An equation (b) Non-unequality
(c) Identity (d) Inequality

Q.2 ax + by < C is linear inequality in variables: *(Rawalpindi Board 2013)*
(a) 2 (b) 3
(c) 1 (d) 0

Q.3 The solution of ax + by < C is: *(Sargodha Board 2013)*
(a) Closed half plane (b) Open half plane
(c) Parabola (d) Hyperbola

Q.4 (2, 1) is the solution of the inequality: *(Sahiwal Board 2013)*
(a) $x + y < 5$ (b) $x + y > 5$
(c) $x + y = 5$ (d) $x - y > 5$

Q.5 (1, 0) is not solution of inequality: *(Multan Board 2013 G-I)*
(a) $7x + 2y < 8$ (b) $x - 3y < 0$
(c) $3x + 5y < 7$ (d) $3x + 5y \leq 3$

Q.6 (1,2) is the solution of: *(Multan Board 2013 G-I)*
(a) $x + y > 0$ (b) $x + y < 0$
(c) $x + y = 0$ (d) $x - y = 0$

Q.7 Which one is a solution of inequality $2x + 3y < 0$? *(Rawalpindi Board 2014)*
(a) (−1, −2) (b) (1, 2)
(c) (2, 3) (d) (0, 1)

Q.8 (1, 3) is in the solution of region: *(Bahawalpur Board 2013 G-I)*
(a) $x + y > 0$ (b) $x + y < 0$
(c) $x + y = 2$ (d) $x - y = 0$

Q.9 The inequality $2x + 3y < 5$ is satisfied by point: *(Multan Board 2014 G-I)*
(a) (1, 1) (b) (−2, 1)
(c) (1, 2) (d) (−2, 3)

Q.10 x = 0 is in the solution of the inequality: *(Multan Board 2014 G-II)*
(a) $2x + 1 > 0$ (b) $2x + 1 < 0$
(c) $2x + 1 \leq 0$ (d) $2x - 1 > 0$

Q.11 x = −5 is in the solution of: *(D.G.K Board 2014 G-I)*
(a) $x - 4 > 0$ (b) $2x + 3 < 0$
(c) $x + 4 > 0$ (d) $x > 0$

Q.12 A vertical line divides a plane into ———— half planes: *(D.G.K Board 2014 G-II, D.G.K Board 2015 G-I)*
(a) Upper and lower (b) Upper and right
(c) Left and right (d) Left and lower

Q.13 (2, 1) is in the solution of inequality: *(D.G.K Board 2010, Sahiwal Board 2014)*
(a) $2x + y \geq 6$ (b) $x - y > 1$
(c) $3x + 5y < 7$ (d) $2x + y \leq 6$

Q.14 x = 0 is not in the solution of inequality: *(D.G.K Board 2013 G-I, D.G.K Board 2015 G-II)*
(a) $2x + 3 > 0$ (b) $x + 4 > 0$
(c) $x + 5 > 0$ (d) $2x + 3 < 0$

Q.15 (⅛, 0) is in the solution of inequality: *(D.G.K Board 2013 G-II)*
(a) $7x + 2y > 0$ (b) $x - 3y > 0$
(c) $3x + 5y > 7$ (d) $3x + 5y < 7$

Q.16 x = −1 is solution of inequality: *(D.G.K Board 2013 G-II)*
(a) $2x + 3 \leq 0$ (b) $2x + 3 > 0$
(c) $x + 5 < 0$ (d) $x > 0$

Q.17 (−2, 3) lies in the solution region of: *(D.G.K Board 2012 G-I)*
(a) $2x - 3y \geq 1$ (b) $x + 2y \geq 7$
(c) $x + 3y \geq 5$ (d) $2x - y \geq 3$

Q.18 The inequality $2x + 3y < 5$ is satisfied by point: *(D.G.K Board 2011)*
(a) (1, 1) (b) (2, 1)
(c) (−2, 1) (d) (−2, 3)

Q.19 (0, 0) is in the solution of inequality: *(D.G.K Board 2010)(Sahiwal Board 2017)*
(a) $2x + y > 6$ (b) $x - y > 1$
(c) $3x + 5y > 7$ (d) $2x + y \leq 6$

Q.20 Solution set of inequality $2x - 3 \geq 0$ equals:
(Sahiwal Board 2015)
(a) $\left[\frac{3}{2}, \infty\right[$ (b) $\left[\frac{3}{2}, \infty\right]$
(c) $\left[\frac{2}{3}, \infty\right]$ (d) $\left[\frac{2}{3}, \infty\right]$

Q.21 $x + 2y > 6$ is not satisfied by:
(Multan Board 2015 G-II)
(a) (2, 3) (b) (2, 2)
(c) (3, 2) (d) (3, 3)

Q.22 The associated equation of inequality $x + 2y < 6$ is:
(Multan Board 2015 G-I)
(a) $x + 2y = 6$ (b) $x - 2y = 6$
(c) $x + 2y = -6$ (d) $x - 2y = -6$

Q.23 The point (1, 2) satisfies inequality:
(Multan Board 2011 G-II)
(a) $x + 2y > 3$ (b) $x - 2y > 3$
(c) $x - 2y \geq 3$ (d) $x + 2y < 3$

Q.24 The point (−1, 2) satisfied the inequality:
(Multan Board 2011 G-I)
(a) $x - y > 4$ (b) $x - y \geq 4$
(c) $x + y < 4$ (d) $x + y > 4$

Q.25 Which one satisfies the inequality $x + 2y < 6$?
(Lahore Board 2014 G-II)
(a) (4, 1) (b) (1, 3)
(c) (1, 4) (d) (3, 1)

Q.26 $x = 4$ is the solution of inequality:
(Lahore Board 2014 G-II)
(a) $x + 3 > 0$ (b) $x - 3 < 0$
(c) $-2x + 3 > 0$ (d) $x + 3 < 0$

Q.27 (1, 0) is the solution of inequality:
(Lahore Board 2015 G-I)
(a) $7x + 2y < 5$ (b) $x - 3y < 0$
(c) $3x + 5y < 6$ (d) $-3x + 5y > 2$

Q.28 The graph of the inequality $ax + by < c$ is:
(Lahore Board 2013 G-II)
(a) Circle (b) Parabola
(c) Straight line (d) Half plane

Q.29 The non-negative inequalities are called:
(Lahore Board 2013 G-I)
(a) Parameters (b) Constants
(c) Decision variables (d) Vertices

Q.30 (3, 2) is not in the solution of inequality:
(Gujranwala Board 2012)(Sahiwal Board 2018)
(a) $x + y > 2$ (b) $3x + 5y > 8$
(c) $3x + 7y > 3$ (d) $3x - 7y < 3$

Q.31 Solution of inequality $x + 2y < 6$ is:
(Lahore Board 2012 G-II)
(a) (1, 1) (b) (1, 3)
(c) (1, 4) (d) (1, 5)

Q.32 The inequality $ax + by \leq c$ when $a = 0$ represents ——— half plane: *(Lahore Board 2012 G-I)*
(a) Left or right (b) Upper or lower
(c) Open (d) None

Q.33 Point (1, 2) lies in the solution region of the inequality: *(Rawalpindi Board 2018)*
(Faisalabad Board 2016, 2018)
(a) $2x + y > 5$ (b) $x + 3y > 5$
(c) $2x + y < 3$ (d) $2x + y > 6$

Q.34 $x = -3$ is the solution of the inequality:
(Gujranwala Board 2016)
(a) $2x - 1 > 0$ (b) $2x + 1 > 0$
(c) $x + 4 < 0$ (d) $2x - 1 < 0$

Q.35 $2x + 3y < 0$ is: *(Sargodha Board 2016)*
(a) An equation (b) Inequality
(c) Identity (d) None of these

Q.36 The solution set of $x < 4$ =
(Rawalpindi Board 2016)
(a) $0 < x < 4$ (b) $10 < x < 15$
(c) $-\infty < x < 4$ (d) $4 < x < \infty$

Q.37 Solution set of inequality $2x < 3$ is:
(Lahore Board 2016 G-I)
(a) $\left(-\infty, \frac{3}{2}\right)$ (b) $\left(\frac{3}{2}, \infty\right)$
(c) $(-\infty, \infty)$ (d) $\left(\frac{-3}{2}, \frac{3}{2}\right)$

Q.38 In-equalities are expressed by the symbols:
(a) $=, <, >, \leq, \geq$ (b) $\neq, <, >, \leq, \geq$
(c) $<, >, \leq, \geq$ (d) None of these

Q.39 $ax \leq b$ is an inequality of one variable:
(a) One variable (b) Three variable
(c) Two variable (d) None of these

Q.40 $ax + by \geq c$ is an inequality of two variables:
(a) One variable (b) Three variable
(c) Two variable (d) None of these

Q.41 The operation ——— a constant to each side of inequality will not affect the order (or sense) of inequality:
(a) Adding (b) Dividing
(c) Multiplying (d) None of these

Q.42 Non-vertical lines divide the plane into ---------- half plane.
(a) Upper and Lower (b) Many
(c) Left and right (d) None of these

Q.43 There are ---------- ordered pairs that satisfy the inequality $ax + by > c$.
(a) Finitely many (b) Two
(c) Infinitely many (d) None of these

Q.44 The graph of linear equation of the form $ax + by = c$ is dividing the plane into two disjoint regions as ----------, where a, b and c are constants and a, b are not both zero.
(a) $ax + by \leq c$ and $ax + by \geq c$
(b) $ax + by < c$ and $ax + by > c$
(c) $ax + by < c$ and $ax + by \geq c$
(d) $ax + by \leq c$ and $ax + by > c$

Q.45 There are --------- ordered pairs that satisfy the inequality $ax + by \leq c$.
(a) Finitely many (b) Two
(c) Infinitely (d) None of these

Q.46 The linear equation ---------- is called the associated or corresponding equation of the inequality $ax + by < c$.
(a) $ax + by \geq c$ (b) $ax + by = c$
(c) $ax + by \leq c$ (d) $ax + by > c$

Q.47 $x = c$ is a vertical line parallel to ---------.
(a) x-axis (b) y-axis may be
(c) y-axis (d) None of these

Q.48 The inequality $x < a$ is the open half plane to the ---------- of the boundary line $x = a$.
(a) Above (b) Left
(c) Below (d) Right

Q.49 The inequality $y > b$ is the open half plane to the ---------- of the boundary line $y = b$.
(a) Above (b) Left
(c) Below (d) Right

Q.50 $x = a$ is a vertical line perpendicular to ---------
(a) x-axis (b) x-axis may be
(c) y-axis (d) None of these

Q.51 The system of ---------- involved in the problem concerned is called problem constraints.
(a) Linear equalities
(b) Equations
(c) Linear inequalities
(d) None of these

Q.52 There are ------- ordered pairs that satisfy the inequality $ax + by < c$.
(a) Finitely many (b) Two
(c) Infinitely many (d) None of these

Q.53 $y = b$ is a horizontal line parallel to ----------.
(a) x-axis (b) x-axis may be
(c) y-axis (d) None of these

Q.54 The inequality $y \geq b$ is the closed half plane to the ---------- of the boundary line $y = b$.
(a) Above (b) Left
(c) Below (d) Right

Q.55 The inequality $y \leq b$ is the closed half plane to the ---------- of the boundary line $y = b$.
(a) Above (b) Left
(c) Below (d) Right

Q.56 $-\infty < x < \frac{3}{2}$ is the solution set of the inequality.
(a) $x = \frac{3}{2}$ (b) $x > \frac{3}{2}$
(c) $x \neq \frac{3}{2}$ (d) $x < \frac{3}{2}$

Q.57 The ordered pair ---------- is a solution of the inequality $x + 2y < 6$.
(a) (3, 3) (b) (1, 1)
(c) (4, 4) (d) None of these

Q.58 The solution set of an inequality $ax + by < 0$ is.
(Sargodha Board 2017)
(a) Open half plane (b) Closed half plane
(c) Circle (d) Parabola

Q.59 Solution region of inequality $2x + y > 2$ lies in.
(D.G.K Board 2017 G-I)
(a) 1st quadrant (b) 2nd quadrant
(c) Towards origin (d) Outwards the origin

Q.60 $ax + b < c$ is. *(Lahore Board 2017 G-II)*
(a) Linear inequality (b) Identity
(c) Equation (d) Not inequality

Q.61 (1, 0) is solution if inequality.
(Lahore Board 2017 G-I)
(a) $9x + 2y < 8$ (b) $-x + 3y < 0$
(c) $3x + 5y < 6$ (d) $3x + 5y > 4$

Q.62 (2, 1) not is in the solution of inequality.
(Multan Board 2017 G-I)
(a) $2x + y \geq 0$ (b) $x - y > 1$
(c) $3x + 5y < 7$ (d) $2x + y \leq 6$

Q.63 (0, 0) lies in the solution set of inequality.
(Rawalpindi Board 2017 G-I)
(a) $x + 2y \leq 0$ (b) $x + 2y \geq 10$
(c) $x + 2y \geq 1$ (d) $x - 2y \geq 10$

Q.64 If $x = -3$ satisfies.
(Rawalpindi Board 2017 G-II)
(a) $x + 3 > 2$ (b) $x + 3 > -2$
(c) $3x > 0$ (d) $x + 2 > 5$

Q.65 (0, 0) is the solution of inequality.
(Sahiwal Board 2017)
(a) $7x + 2y > 3$ (b) $x - 3y > 0$
(c) $x + 2y < 6$ (d) $x - 3y < 0$

Q.66 System of linear inequalities involved in the problem is called. *(Lahore Board 2018 G-II)*
(a) Co-efficient (b) Solution
(c) Problem constant (d) Boundaries

Q.67 The graph of $2x > 3$ lies in.
(Multan Board 2018 G-I)
(a) Upper half plane (b) Lower half plane
(c) Left half plane (d) Right half plane

Q.68 _____ point satisfy $x - y < 2$.
(Multan Board 2018 G-II)
(a) (3, 1) (b) (−1, 1)
(c) (1, −1) (d) (0, −2)

Q.69 $x = 0$ is not in the solution of inequality.
(Sargodha Board 2018)
(a) $2x + 3 > 0$ (b) $x + 4 > 0$
(c) $x + 5 > 0$ (d) $2x + 3 < 0$

Q.70 The Graph of the inequality $ax + by < c$ is:
(Bahawalpur Board 2019)
(a) Circle (b) Parabola
(c) Straight Line (d) Half Plane

Q.71 The solution of the inequality $2x + y < 5$ is:
(Faisalabad Board 2019)
(a) (1 , 2) (b) (2 , 1)
(c) (2 , 3) (d) (5 , 9)

Q.72 (1 , 0) is not the solution of the inequality:
(Faisalabad Board 2019 G-II)
(a) $x - 3y < 0$ (b) $7x + 2y < 8$
(c) $3x + 5y < 7$ (d) $4x - 3y < 9$

Q.73 (1, −3) is in the solution of region
(Gujranwala Board 2019 G-I)
(a) $x + y > 0$ (b) $x + y < 0$
(c) $x + y = 0$ (d) $x - y = 0$

Q.74 (1, 0) is the solution of the inequality
(Gujranwala Board 2019 G-II)
(a) $7x + 2y < 8$ (b) $x - 3y < 0$
(c) $10x + 5y < 6$ (d) $-3x + 5y > 2$

Q.75 The solution of the inequality $x + 2y < 6$ is:
(Lahore Board 2019)
(a) (1 , 1) (b) (1 , 3)
(c) (1 , 4) (d) (1 , 5)

Q.76 If $3x + 2y \leq 6$, point does not satisfy:
(Multan Board 2019)
(a) (1 , 0) (b) (0 , 1)
(c) (0 , 0) (d) (3 , 2)

Q.77 $x = 0$ is he solution of the inequality.
(Rawalpindi Board 2019)
(a) $2x + 1 > 0$ (b) $2x + 1 < 0$
(c) $2x + 1 \leq 0$ (d) $2x - 1 < 0$

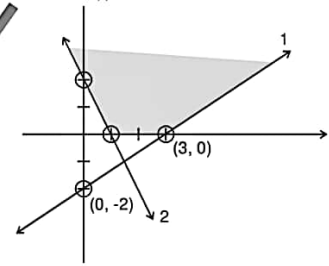
Q.78 (1 , 1) is solution of *(Sahiwal Board 2019)*
(a) $x + y < 1$ (b) $2x + y < 1$
(c) $2x - y < 1$ (d) $x - y < 1$

EXERCISE 5.2

LONG QUESTIONS

Q.1 Graph the feasible region subject to the following constraints:
$2x - 3y \leq 6 : 2x + y \geq 2 : x \geq 0, y \geq 0$
(Lahore Board 2013)(Sargodha Board 2016)
(Lahore Board 2014, 2017 G-II)

Sol: $2x - 3y \leq 6$
$2x + y \geq 2$
$x \geq 0 , y \geq 0$
The corresponding equations are
$2x - 3y = 6$ ... (i)
$2x + y = 2$ ... (ii)
$x = 0$ ... (iii)
$y = 0$ ... (iv)
As $x \geq 0 , y \geq 0$, required region lies in first quadrant
For equation (i) $2x - 3y = 6$
Put $x = 0 \Rightarrow -3y = 6 \Rightarrow y = -2$
$y = 0 \Rightarrow 2x = 6 \Rightarrow x = 3$
Draw a straight line joining (0, −2) and (3, 0).
Put (0, 0) in $2x - 3y \leq 6$, we get $0 \leq 6$, it is true. So, we shade above line (i)

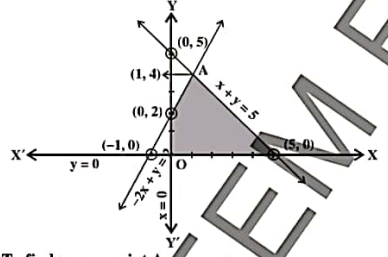


For equation (ii) , $2x + y = 2$
Put $x = 0 \Rightarrow y = 2$
$y = 0 \Rightarrow 2x = 2 \Rightarrow x = 1$
We draw a straight line (ii) joining (0, 2) and (1, 0). Put (0, 0) in $2x + y \geq 2$ we get $0 \geq 2$, it is false. So we shade above line (ii).

Q.2 Graph the feasible of inequalities and find the corner point.
$x + y \leq 5 , -2x + y \leq 2 , x \geq 0 , y \geq 0$
(D.G Khan Board 2017 G-I)

Sol: Given that:
$x + y \leq 5 , -2x + y \leq 2 , x \geq 0, y \geq 0$

$x + y \leq 5$	$-2x + y \leq 2$	$x \geq 0 \mid y \geq 0$
Associated equation	Associated equation	Solution region lies in 1st quadrant
$x + y = 5$ … (1)	$-2x+y=2$ …(2)	
Put $x = 0$ in eq (1)	Put $x = 0$, in eq (2)	
$0 + y = 5$	$-2(0) + y = 2$	
$y = 5$	$y = 2$	
$\therefore$ (0, 5)	$\therefore$ (0, 2)	
Put $y = 0$ in eq (1)	Put $y = 0$, in eq (2)	
$x + 0 = 5$	$-2x + 0 = 2$	
$x = 5$	$-2x = 2$	
$\therefore$ (5, 0)	$x = -1$	
Test point	$\therefore$ (−1, 0)	
Put $x = 0$, $y = 0$ in	Test point	
$x + y < 5$	Put $x = 0$, $y = 0$ in	
$0 + 0 < 5$	$2x + y < 2$	
$0 < 5$ (True)	$0 + 0 < 2$	
Solution region lies towards the origin side	$0 < 2$ (True) Solution region lies towards the origin side	



To find corner point A:

$x + y = 5$ …… (i)

$-2x + y = 2$ …… (ii)

eq (i) − eq (ii)

$x + y = 5$

$-2x + y = 2$

$+ \quad - \quad -$

$3x = 3 \Rightarrow x = 1$

Putting value of x in eq. (i):

$1 + y = 5 \Rightarrow y = 5 - 1 = 4$

$\therefore$ A(1, 4)

Hence (0, 0), (1, 4), (0, 2) and (5, 0) are the corner points.

Q.3 Graph the feasible region and find the corner points of linear inequalities.

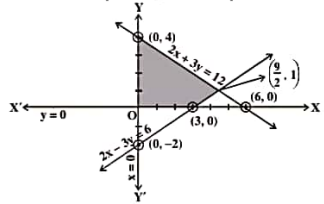
$2x - 3y \leq 6$, $2x + 3y \leq 12$. $x \geq 0$, $y \geq 0$.

(Lahore Board 2015 G-I, G-II)(Gujranwala Board 2013) (Sahiwal Board 2015)(Gujranwala Board 2016, 2012)

Sol: Given that:

$2x - 3y \leq 6$, $2x + 3y \leq 12$, $x \geq 0$, $y \geq 0$

$2x - 3y \leq 6$	$2x + 3y \leq 12$	$x \geq 0 \mid y \geq 0$
Associated equation	Associated equation	Solution region lies in 1st quadrant
$2x - 3y = 6$ … (1)	$2x+3y=12$…(2)	
Put	Put	
$x = 0$, in eq (1)	$x = 0$, in eq (2)	
$2(0) - 3y = 6$	$2(0) + 3y = 12$	
$-3y = 6$	$3y = 12$	
$y = -2$	$y = 4$	
$\therefore$ (0, −2)	$\therefore$ (0, 4)	
Put	Put	
$y = 0$, in eq (1)	$y = 0$, in eq (2)	
$2x - 3(0) = 6$	$2x + 3(0) = 12$	
$2x = 6$	$2x = 12$	
$x = 3$	$x = 6$	
$\therefore$ (3, 0)	$\therefore$ (6, 0)	
Test point	Test point	
Put $x = 0$, $y = 0$ in	Put	
$2x - 3y < 6$	$x = 0$, $y = 0$ in	
$0 - 0 < 6$	$2x + 3y < 12$	
$0 < 6$ (True)	$0 + 0 < 12$	
Solution region lies towards the origin side	$0 < 12$ (True) Solution region lies towards the origin side	



To find corner point A:

$2x - 3y = 6$ …… (i)

$2x + 3y = 12$ …… (ii)

Adding eq. (i) and (ii):

$2x - 3y = 6$

$2x + 3y = 12$

$4x = 18 \Rightarrow x = \frac{18}{4} = \frac{9}{2}$

Putting the value of x in eq. (i):

$2\left(\frac{9}{2}\right) - 3y = 6 \Rightarrow 9 - 3y = 6$

$-3y = 6 - 9 \Rightarrow -3y = -3 \Rightarrow y = 1$

$\therefore A\left(\frac{9}{2}, 1\right)$

Hence $\left(\frac{9}{2}, 1\right)$, (0, 4), (0, 0) and (3, 0) are corner points.

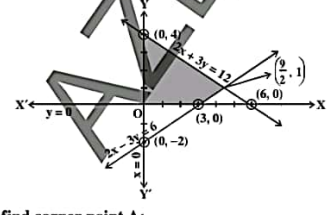
Q.4 Graph the feasible region and also find the corner points $2x - 3y \leq 6, 2x + 3y \leq 12, x \geq 0, y \geq 0$

(Gujranwala Board 2012, 2016)
(Lahore Board 2015 G-I)

Sol: Given that:

$2x - 3y \leq 6$, $2x + 3y \leq 12$, $x \geq 0$, $y \geq 0$

$2x - 3y \leq 6$	$2x + 3y \leq 12$	$x \geq 0 \mid y \geq 0$
Associated equation	Associated equation	Solution region lies in 1st quadrant
$2x - 3y = 6$ … (1)	$2x+3y=12$…(2)	
Put	Put	
$x = 0$, in eq (1)	$x = 0$, in eq (2)	
$2(0) - 3y = 6$	$2(0) + 3y = 12$	
$-3y = 6$	$3y = 12$	
$y = -2$	$y = 4$	
$\therefore (0, -2)$	$\therefore (0, 4)$	
Put	Put	
$y = 0$, in eq (1)	$y = 0$, in eq (2)	
$2x - 3(0) = 6$	$2x + 3(0) = 12$	
$2x = 6$	$2x = 12$	
$x = 3$	$x = 6$	
$\therefore (3, 0)$	$\therefore (6, 0)$	
Test point	Test point	
Put $x = 0$, $y = 0$ in	Put	
$2x - 3y < 6$	$x = 0$, $y = 0$ in	
$0 - 0 < 6$	$2x + 3y < 12$	
$0 < 6$ (True)	$0 + 0 < 12$	
Solution region lies towards the origin side	$0 < 12$ (True) Solution region lies towards the origin side	



To find corner point A:

$2x - 3y = 6$ …… (i)

$2x + 3y = 12$ …… (ii)

Adding eq. (i) and (ii):

$2x - 3y = 6$

$2x + 3y = 12$

$4x = 18 \Rightarrow x = \frac{18}{4} = \frac{9}{2}$

Putting the value of x in eq. (i):

$2\left(\frac{9}{2}\right) - 3y = 6 \Rightarrow 9 - 3y = 6$

$-3y = 6 - 9 \Rightarrow -3y = -3 \Rightarrow y = 1$

$\therefore A\left(\frac{9}{2}, 1\right)$

Hence $\left(\frac{9}{2}, 1\right)$, (0, 4), (0, 0) and (3, 0) are corner points.

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 The solution region restricted to 1st quadrant is: *(Multan Board 2012 G-II)*

(a) Solution region (b) Feasible solution
(c) Feasible region (d) Optimal origin

Q.2 For inequalities $2x + y \leq 10$ and $x + 4y \leq 12$, the corner point: *(Multan Board 2012 G-I)*

(a) (5, 10) (b) (12, 3)
(c) (4, 2) (d) (10, 12)

Q.3 A point of a solution region where two of its boundary lines intersects is called a ---------- point of the solution region.

(a) Maximum (b) Corner
(c) Minimum (d) None of these

Q.4 The feasible region is ---------- if it can easily be enclosed within a circle.

(a) Bounded (b) Exist
(c) Unbounded (d) None of these

Q.5 For different values of k, the equation $4x + 5y = k$ represents lines ---------- to the line $4x + 5y = 0$.

(a) Perpendicular (b) Parallel
(c) Equal (d) None of these

EXERCISE 5.3

SHORT ANSWERS TO THE QUESTIONS

Q.1 Define objective function and optimal solution.
(Multan Board 2013 G-II)(Lahore Board 2017 G-II)
(Lahore, Rawalpindi Board 2017 G-II)
(Sargodha Board 2013, 20162017, 2018)
(A.J.K Board 2017) (Gujranwala Board 2016, 2017)
(Faisalabad Board 2017, 2019 G-I)
(D.G Khan Board 2017 G-I)(Sahiwal Board 2019)

Ans. Objective function: A function, which is to be maximized or minimized is called an objective function.

Optimal solution: The feasible solution which maximizes or minimizes the objective function is called optimal solution.

LONG QUESTIONS

Q.1 Maximize $f(x, y) = 2x + 5y$ subject to constraints: $2y - x \le 8$, $x - y \le 4$, $x \ge 0$, $y \ge 0$
(Faisalabad Board 2013, 2019 G-I)
(Rawalpindi Board 2013)(Bahawalpur Board 2016)
(Lahore Board 2013 G-II)

Sol:

$2y - x \le 8$

Associated equation

$2y - x = 8$ ..... (1)

Put $y = 0$ in equation (1)

$0 - x = 8 \Rightarrow x = -8$

$\Rightarrow$ A $(-8, 0)$

Put $x = 0$ in equation (1)

$2y - 0 = 8 \Rightarrow y = 4$

$\Rightarrow$ B $(0, 4)$

Test Point

Put $x = 0$, $y = 0$ in $2y - x < 8$

$0 - 0 < 8 \Rightarrow 0 < 8$ (true)

Solution region lies below the line $2y - x = 8$

$x - y \le 4$

Associated equation

$x - y = 4$ ..... (2)

Put $y = 0$ in equation (2)

$x - 0 = 4 \Rightarrow x = 4$

$\Rightarrow$ C $(4, 0)$

Put $x = 0$ in equation (2)

$0 - y = 4 \Rightarrow y = -4$

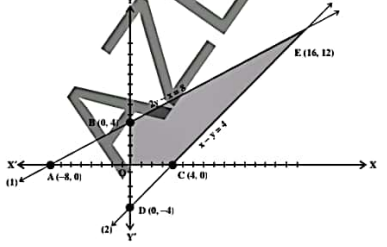
$\Rightarrow$ D $(0, -4)$

Test Point

Put $x = 0$, $y = 0$ in $x - y < 4$

$0 - 0 < 4 \Rightarrow 0 < 4$ (true)

Solution region lies below the line $x - y = 4$



To find corner point E

eq (1) + eq (2)

$2y - x = 8$

$x - y = 4$

$y = 12$

Put $y = 12$ in eq (2)

$x - 12 = 4 \Rightarrow x = 16$

E (16, 12)

Corner Points	f (x, y) = 2x + 5y	
(0, 0)	= 0 + 0	= 0
(4, 0)	= 8 + 0	= 8
(16, 12)	= 32 + 60	= 92
(0, 4)	= 0 + 20	= 20

The maximum value of f in 92 at the corner point (16 , 12)

Q.2 Maximize $f(x, y) = x + 3y$ subject to the constrains. $2x + 5y \le 30$; $5x + 4y \le 20$; $x \ge 0$; $y \ge 0$
(Multan 2015, 2016 G-I)(Sahiwal Board 2013)
(Rawalpindi Board 2014)(Gujranwala Board 2014)
(D.G.K Board 2014 G-I)(Lahore Board 2016, 2017 G-I)
(Sargodha Board 2018)

Sol:

$2x + 5y \le 30$

Associated equation

$2x + 5y = 30$ ..... (1)

Put $y = 0$ in equation (1)

$2x + 0 = 30 \Rightarrow x = 15$

$\Rightarrow$ A $(15, 0)$

Put $x = 0$ in equation (1)

$0 + 5y = 30 \Rightarrow y = 6$

$\Rightarrow$ B $(0, 6)$

Test Point

Put $x = 0$, $y = 0$ in $2x + 5y < 30$

$0 - 0 < 30 \Rightarrow 0 < 30$ (true)

Solution region lies below the line $2x + 5y = 30$

$5x + 4y \le 20$

Associated equation

$5x + 4y = 20$ ..... (2)

Put $y = 0$ in equation (2)

$5x + 0 = 20 \Rightarrow x = 4$

$\Rightarrow$ C $(4, 0)$

Put $x = 0$ in equation (2)

$0 + 4y = 20 \Rightarrow y = 5$

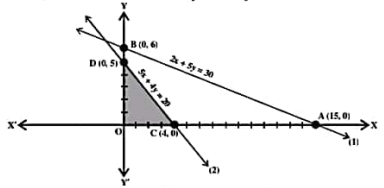
$\Rightarrow$ D $(0, 5)$

Test Point

Put $x = 0$, $y = 0$ in $x - y < 4$

$0 - 0 < 4 \Rightarrow 0 < 4$ (true)

Solution region lies below the line $5x + 4y = 20$



Corner Points	f (x, y) = x + 3y	
(0, 0)	= 0 + 0	= 0
(4, 0)	= 4 + 0	= 4
(0, 5)	= 0 + 15	= 15

The maximum value of f is 15 at the corner point (0,5)

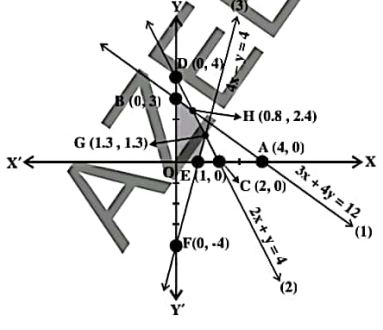
Q.3 Maximize f(x, y) = 2x + 3y subject to the constraints $3x + 4y \le 12$, $2x + y \le 4$, $4x - y \le 4$

$x \ge 0$, $y \ge 0$. *(D.G.K Board 2014 G-II)*

(Multan Board 2018 G-II)(Sargodha Board 2019)

Sol:

$3x + 4y \le 12$	$2x + y \le 4$	$4x - y \le 4$
Associated equation	Associated equation	Associated equation
$3x + 4y = 12$ ...(1)	$2x + y = 4$ ...(2)	$4x - y = 4$ ...(3)
Put y = 0 in equation (1)	Put y = 0 in equation (1)	Put y = 0 in equation (1)
$3x + 0 = 12$	$2x + 0 = 4$	$4x - 0 = 4$
$\Rightarrow x = 4$	$\Rightarrow x = 2$	$\Rightarrow x = 1$
A (4,0)	C(2,0)	E(1,0)
Put x = 0 in equation (1)	Put x = 0 in equation (1)	Put x = 0 in equation (1)
$0 + 4y = 12$	$0 + y = 4$	$0 - y = 4$
$\Rightarrow y = 3$	$\Rightarrow y = 4$	$\Rightarrow y = -4$
B(0,3)	D(0,4)	F(0,–4)
Test Point	Test point	Test point
Put x = 0, y = 0 in $3x = 4y < 12$	Put x = 0, y = 0 in $2x + y < 4$	Put x = 0, y = 0 in $4x - y < 4$
$0 + 0 < 12$	$0 - 0 < 4$	$0 - 0 < 4$
$\Rightarrow 0 < 12$ (true)	$\Rightarrow 0 < 4$ (true)	$\Rightarrow 0 < 4$ (true)
Solution region lies below the line $3x = 4y = 12$	Solution region lies below the line $2x + y = 4$	Solution region lies below the line $4x = y = 4$



To find corner point G and H

eq(2) + eq(3)	eq(1) – 4eq(2)
$2x + y = 4$	$3x + 4y = 12$
$4x - y = 4$	$\pm 8x \pm 4y = \pm 16$
$6x = 8 \Rightarrow x = \frac{8}{6} = \frac{4}{3}$	$-5x = -4 \Rightarrow x = \frac{4}{5}$
put $x = \frac{4}{3}$ in eq(3)	Put $x = \frac{4}{5}$ in eq(2)
$4\left(\frac{4}{3}\right) - y = 4$	$2\left(\frac{4}{5}\right) + y = 4$
$\Rightarrow y = \frac{16}{3} - 4 = \frac{4}{3}$	$\Rightarrow y = 4 - \frac{8}{5} = \frac{12}{5}$
$G\left(\frac{4}{3}, \frac{4}{3}\right)$	$H\left(\frac{4}{5}, \frac{12}{5}\right)$

corner points	f(x,y) = 2x + 3y
(0,0)	= 0 + 0 = 0
(1,0)	= 2 + 0 = 2
$\left(\frac{4}{3}, \frac{4}{3}\right)$	$= \frac{8}{3} + \frac{12}{3} = \frac{20}{3}$
$\left(\frac{4}{5}, \frac{12}{5}\right)$	$= \frac{8}{5} + \frac{36}{5} = \frac{44}{5}$
(0,3)	= 0 + 9 = 9

The maximum value of f is 9 at the corner point (0,3)

Q.4 Minimize x = 2x + y subject to the constraints.

$x + y \ge 3$, $7x + 5y \le 35$, $x \ge 0$, $y \ge 0$.

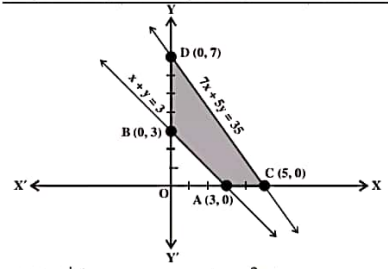
(Lahore Board 2014, 2018 G-I, 2019 G-II)

(Multan Board 2012, 2018 G-II)

(Faisalabad Board 2017)(Sahiwal Board 2019)

Sol:

$x + y \ge 3$	$7x + 5y \le 35$
Associated equation	Associated equation
$x + y = 3$ ...(1)	$7x = 5y = 35$ ...(2)
Put y = 0 in equation (1)	Put y = 0 in equation (2)
$x + 0 = 3 \Rightarrow x = 3$	$7x + 0 = 35 \Rightarrow x = 5$
A (3,0)	C(5,0)
Put x = 0 in equation (1)	Put x = 0 in equation (2)
$0 + y = 3 \Rightarrow y = 3$	$0 + 5y = 35 \Rightarrow y = 7$
B(0,3)	D(0,7)
Test Point	Test Point
Put	Put
x = 0, y = 0 in $x + y > 3$	x = 0,y = 0in $7x + 5y < 35$
$0 + 0 > 3 \Rightarrow 0 > 3$ (false)	$0 + 0 < 35 \Rightarrow 0 < 35$(true)
Solution region lies above the line $x + y = 3$	Solution region lies below the line $7x + 5y = 35$



corner points	z = 2x + y
(3,0)	= 6 + 0 = 6
(0,3)	= 0 + 3 = 3
(5,0)	= 10 + 0 = 10
(0,7)	= 0 + 7 = 7

The minimum value is 3 at the corner point (0,3)

Q.5 Minimize f(x, y) = 3x + y subject to the constrains.

$3x + 5y \geq 15$

$x + 3y \geq 9 \qquad x \geq 0, y \geq 0$

(Sargodha Board 2013)(Multan 2016 G-II) (Sahiwal, Gujranwala Board 2018) (D.G.Khan Board 2017 G-II)(Bahawalpur Board 2019)

Sol: $x + 6y \geq 9$

Associated equation

$x + 6y = 9$ ..... (1)

Put y = 0 in equation (1)

$x + 0 = 9 \Rightarrow x = 9$

$\Rightarrow$ A (9, 0)

Put x = 0 in equation (1)

$0 + 6y = 9 \Rightarrow y = \frac{9}{6} = \frac{3}{2}$

$\Rightarrow B\left(0, \frac{3}{2}\right)$

Test Point

Put x = 0 , y = 0 in x + 6y < 9

0 + 0 > 9 $\Rightarrow$ 0 > 9 (false)

Solution region lies above the line x + 6y = 9

$3x + 5y \geq 15$

Associated equation

$3x + 5y = 15$ ..... (2)

Put y = 0 in equation (2)

$3x + 0 = 15 \Rightarrow x = 5$

$\Rightarrow$ C (5, 0)

Put x = 0 in equation (2)

$0 + 5y = 15 \Rightarrow y = 3$

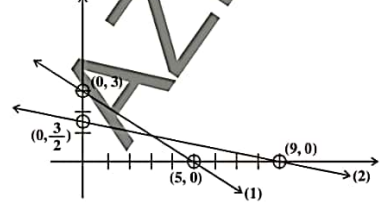
$\Rightarrow$ D (0, 3)

Test Point

Put x = 0 , y = 0 in 3x + 5y > 15

0 + 0 > 15 $\Rightarrow$ 0 > 15 (false)

Solution region lies above the line 3x + 5y = 15



To find corner point E

3eq (1) – eq(2)

$3x + 18y = 27$

$\pm 3x \pm 5y = \pm 15$

$13y = 12 \Rightarrow y = \frac{12}{13}$

Put $y = \frac{12}{13}$ in eq (1)

$x + 6\left(\frac{12}{13}\right) = 9 \Rightarrow x = 9 - \frac{72}{13} = \frac{45}{13}$

$E\left(\frac{45}{13}, \frac{12}{13}\right)$

Corner points	z = 3x + y
(0, 3)	= 0 + 3 = 3
$\left(\frac{45}{13}, \frac{12}{13}\right)$	$= \frac{135}{13} + \frac{12}{13} = \frac{147}{13}$
(9, 0)	= 27 + 0 = 27

The minimum value of z is 3 at the corner point (0, 3).

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 A function which is to be maximized or minimized is called: *(Lahore Board 2015 G-II) (Bahawalpur Board 2018)*

(a) Exponential function (b) Linear function
(c) Quadratic function (d) Objective function

Q.2 The feasible solution which maximize or minimize the objective function is called the: *(Gujranwala Board 2013)(Lahore Board 2018 G-I)*

(a) Feasible region (b) Optimal solution
(c) Convex region (d) Feasible solution set

Q.3 The feasible solution, which maximizes or minimizes the objective function, is called the ----------.

(a) Maximum solution (b) Optimal solution
(c) Minimum solution (d) None of these

Q.4 A function which is to be maximized or minimized in called. *(Faisalabad Board 2017)*

(a) Subjective function (b) Objective function
(c) Qualitative function (d) Quantitative function

Q.5 To find optimal solution we evaluate the objective function at. *(Gujranwala Board 2018)*

(a) One point (b) Origin
(c) Some points (d) Corner points

Q.6 A function which is to be maximized or minimized is called. *(Sargodha Board 2019)*

(a) Objective function (b) Optimal function
(c) Constant function (d) Polynomial function

Chapter 6

CONIC SECTIONS

EXERCISE 6.1

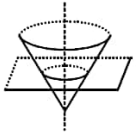
SHORT ANSWERS TO THE QUESTIONS

Q.1 Give original definition of conics.

Ans. The curves obtained by cutting a double right circular cone by a plane are called conics.

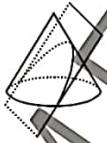
Q.2 Define circle as plane section of a cone.

Ans. If a plane cuts the cone perpendicular to the axis of the cone, then the section is a circle.



Q.3 Define parabola as plane section of a cone.

Ans. If the cutting plane is parallel to the generator of the cone, but intersects its one nappe only, then the section is a parabola.



Q.4 Define ellipse as plane section of a cone.

Ans. If the cutting plane is slightly tilled and cuts only one nappe of the cone, then the section is an ellipse.

Q.5 Define hyperbola as plane section of a cone.

Ans. If the cutting plane is parallel to the axis of the cone and intersects both of its nappes, then the section is a hyperbola.

Q.6 Give definition of circle.

(Faisalabad Board 2019 G-II)

Ans. The path traced by a moving point, at a fixed distance from a fixed point, is called a circle.

Q.7 Write standard equation of the circle.

(Sargodha Board 2019)

Ans. The equation of a circle in standard form is

$$(x-h)^2 + (y-k)^2 = r^2$$

Where (h, k) is the centre and r is the radius of the circle.

Q.8 Show that the equation

$x^2 + y^2 + 2gx + 2fy + c = 0$

represents a circle where g, f and c being constants.

Ans. $x^2 + y^2 + 2gx + 2fy + c = 0$.....(1)

Equation (1) can be written as:

$$x^2 + 2gx + y^2 + 2fy = -c$$

Adding $g^2 + f^2$ on both sides, we have

$$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$$

$$\Rightarrow (x+g)^2 + (y+f)^2 = \left(\sqrt{g^2+f^2-c}\right)^2$$

or $$[x-(-g)]^2 + [y-(-f)]^2 = \left(\sqrt{g^2+f^2-c}\right)^2$$

which is standard form of an equation of a circle with center $(-g, -f)$ and radius $\sqrt{g^2+f^2-c}$.

Thus equation (1) represents a circle.

Q.9 Find an equation of the circle with centre at $(\sqrt{2}, -3\sqrt{3})$ and radius $2\sqrt{2}$

(D.G Khan Board 2014 G-II)(A.J.K Board 2017)

Ans. Equation of the circle with centre at (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

$\therefore$ Equation of the circle with centre at $(\sqrt{2}, -3\sqrt{3})$ and radius $2\sqrt{2}$ is

$$\left(x-\sqrt{2}\right)^2 + \left\{y-\left(-3\sqrt{3}\right)\right\}^2 = \left(2\sqrt{2}\right)^2$$

$$\left(x-\sqrt{2}\right)^2 + \left(y+3\sqrt{3}\right)^2 = 4\times 2$$

$$x^2 - 2\sqrt{2}x + 2 + y^2 + 6\sqrt{3}y + 27 = 8$$

$$x^2 + y^2 - 2\sqrt{2}x + 6\sqrt{3}y + 21 = 0$$

Q.10 Show that $5x^2 + 5y^2 + 24x + 36y + 10 = 0$ represents a circle. Also find its center and radius. *(Gujranwala Board 2013) (Sahiwal Board 2013)(Lahore Board 2015 G-I)*

Ans. $5x^2 + 5y^2 + 24x + 36y + 10 = 0$

$$x^2 + y^2 + \frac{24}{5}x + \frac{36}{5}y + 2 = 0$$

$$x^2 + \frac{24}{5}x + y^2 + \frac{36}{5}y = -2$$

$$x^2 + \frac{24}{5}x + \left(\frac{12}{5}\right)^2 + y^2 + \frac{36}{5}y + \left(\frac{18}{5}\right)$$

$$= -2 + \left(\frac{12}{5}\right)^2 + \left(\frac{18}{5}\right)$$

$$\left(x + \frac{12}{5}\right)^2 + \left(y + \frac{18}{5}\right)^2 = -2 + \frac{144}{25} + \frac{324}{25}$$

$$\left(x + \frac{12}{5}\right)^2 + \left(y + \frac{18}{5}\right)^2 = \frac{-50 + 144 + 324}{25}$$

$$\left(x + \frac{12}{5}\right)^2 + \left(y + \frac{18}{5}\right)^2 = \frac{418}{25}$$

$$\left(x + \frac{12}{5}\right)^2 + \left(y + \frac{18}{5}\right)^2 = \left(\frac{\sqrt{418}}{5}\right)^2$$

Which is equation of circle with centre $\left(\frac{-12}{5}, \frac{-18}{5}\right)$ and radius $\frac{\sqrt{418}}{5}$.

Q.11 Find the centre and radius of the circle with the equation $4x^2 + 4y^2 - 8x + 12y - 25 = 0$ *(Gujranwala Board 2012, 2016, 2019 G-II) (Sargodha Board 2016)(Sargodha Board 2017)*

Ans. $4x^2 + 4y^2 - 8x + 12y - 25 = 0$ ..... (i)

Dividing by 4, we have

$$x^2 + y^2 - 2x + 3y - \frac{25}{4} = 0$$

$$x^2 + y^2 + 2(-1)x + 2\left(\frac{3}{2}\right)y - \frac{25}{4} = 0$$

General form of circle is given by

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ ..... (ii)}$$

Comparing (i) and (ii), we get $g = -1$, $f = \frac{3}{2}$ and $c = \frac{-25}{4}$

Thus the centre of circle is at $(-g, -f) = \left(1, \frac{-3}{2}\right)$

and radius $r = \sqrt{g^2 + f^2 - c} = \sqrt{(-1)2 + \left(\frac{3}{2}\right)^2 - \left(\frac{-25}{4}\right)}$

$$= \sqrt{1 + \frac{9}{4} + \frac{25}{4}} = \sqrt{\frac{19}{2}}$$

Q.12 Find an equation of the circle with ends of a diameter at (–3, 2) and (5, –6). *(Gujranwala Board 2014)(Multan Board 2014 G-I) (Bahawalpur Board 2016)(Lahore Board 2017 G-I)*

Ans. Let ends of diameter of required circle are A(–3, 2) and B(5, –6) and C be the centre of the circle then C will be the midpoint of line $\overline{AB}$.

Centre C is at $\left(\frac{5-3}{2}, \frac{-6+2}{2}\right) = (1, -2)$

Radius $= \frac{1}{2}|\overline{AB}| = \frac{1}{2}\sqrt{(5-(-3))^2 + (-6-2)^2}$

$$= \frac{1}{2}\sqrt{(5+3)^2 + (-8)2}$$

$$= \frac{1}{2}\sqrt{64 + 64} = \frac{\sqrt{128}}{2}$$

$$= \frac{\sqrt{64 \times 2}}{2} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$$

Thus the required equation of the circle is

$$(x-1)^2 + [(y-(-2)]^2 = \left(4\sqrt{2}\right)^2$$

$$(x-1)^2 + (y+2)^2 = 16 \times 2$$

$$x^2 - 2x + 1 + y^2 + 4y + 4 = 32$$

$$x^2 + y^2 - 2x + 4y - 27 = 0$$

Q.13 Find center and radius of circle $5x^2 + 5y^2 + 14x + 12y - 10 = 0$ *(Gujranwala Board 2019)(Sahiwal Board 2019)*

Ans. The given equation can be written as

$$x^2 + y^2 + \frac{14}{5}x + \frac{12}{5}y - 2 = 0$$

Equation of a circle in general form is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Here $2g = \frac{14}{5} \Rightarrow g = \frac{7}{5}$, $2f = \frac{12}{5} \Rightarrow f = \frac{6}{5}$, $c = -2$

Center $(-g, -f) = \left(-\frac{7}{5}, -\frac{6}{5}\right)$

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{\frac{49}{25} + \frac{36}{25} + 2}$$

$$= \sqrt{\frac{49 + 36 + 50}{25}} = \sqrt{\frac{135}{25}} = \sqrt{\frac{27}{5}}$$

LONG QUESTIONS

Q.1 Find equation of a circle passing through the points A(–7, 7), B(5, –1), C(10, 0). *(Faisalabad Board 2013)(Bahawalpur Board 2016) (Sahiwal Board 2018)*

Sol: Let D(h, k) be the center of the circle passing through the points A(–7, 7), B(5, –1), C(10, 0) then

$$|\overline{AD}|^2 = |\overline{BD}|^2$$

$(h+7)^2 + (k-7)^2 = (h-5)^2 + (k+1)^2$

$h^2 + 49 + 14h + k^2 + 49 - 14k$

$= h^2 + 25 - 10h + k^2 + 1 + 2k$

$24h - 16k + 72 = 0 \Rightarrow 8(3h - 2k + 9) = 0$

$3h - 2k + 9 = 0 \quad ---(1)$

And

$|\overline{AD}|^2 = |\overline{CD}|^2$

$(h+7)^2 + (k-7)^2 = (h-10)^2 + (k-0)^2$

$h^2 + 49 + 14h + k^2 + 49 - 14k$

$= h^2 + 100 - 20h + k^2$

$34h - 14k - 2 = 0 \Rightarrow 2(17h - 7k - 1) = 0$

$17h - 7k - 1 = 0 \quad ---(2)$

17 Equation (1) − 3 Equation (2)

$51h - 34k + 153 = 0$

$\pm 51h \mu\ 21k \mu\ 3 = 0$

$-13k + 156 = 0 \quad \Rightarrow k = 12$

Put the value of k = 12 in equation (1)

$3h - 24 + 9 = 0 \Rightarrow 3h = 15 \Rightarrow h = 5$

$\therefore$ D(5, 12)

$r = |AD| = \sqrt{(5+7)^2 + (12-7)^2}$

$= \sqrt{144 + 25} = \sqrt{169} = 13$

Equation of circle is

$(x-h)^2 + (y-k)^2 = r^2$

$(x-5)^2 + (y-12)^2 = (13)^2$

$x^2 + 25 - 10x + y^2 + 144 - 24y = 169$

$x^2 + y^2 - 10x - 24y = 0$

Q.2 Find an equation of a circle passing through A(−3,1) with radius 2 and centre at $2x - 3y + 3 = 0$.

(Gujranwala Board 2013)(Lahore Board 2015 G-II)

Sol: Let C(h, k) be the center of the circle passing through the point A(−3, 1) with radius 2 and center at

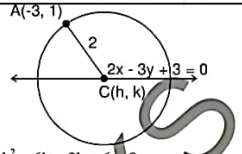
$2x + 3y + 3 = 0$ then

$2h - 3k + 3 = 0 \quad ---(1)$

$|\overline{AC}|^2 = r^2$

$(h+3)^2 + (k-1)^2 = (2)^2$

$h^2 + 9 + 6h + k^2 + 1 - 2k = 4$



$h^2 + k^2 + 6h - 2k + 6 = 0 \quad ---(2)$

From (1) $3k = 2h + 3 \Rightarrow k = \dfrac{2h+3}{3} \quad ---(3)$

Put $k = \dfrac{2h+3}{3}$ in eq. (2)

$h^2 + \left(\dfrac{2h+3}{3}\right)^2 + 6h - 2\left(\dfrac{2h+3}{3}\right) + 6 = 0$

$h^2 + \dfrac{4h^2 + 9 + 12h}{9} + 6h - \left(\dfrac{4h+6}{3}\right) + 6 = 0$

$9h^2 + 4h^2 + 9 + 12h + 54h - 12h - 18 + 54 = 0$

$13h^2 + 54h + 45 = 0 \Rightarrow 13h^2 + 39h + 15h + 45 = 0$

$13h(h+3) + 15(h+3) = 0 \Rightarrow (h+3)(13h+15) = 0$

$\Rightarrow \quad h = -3, \ h = \dfrac{-15}{13}$

Put $h = \dfrac{-15}{13}$ in eq. (3) $k = \dfrac{-\frac{30}{13} + 3}{3} = \dfrac{\frac{-30+39}{13}}{3}$

$k = \dfrac{9}{13 \times 3} = \dfrac{3}{13}$

Put $h = 3$ in eq (3)

$k = \dfrac{2(-3)2 + 3}{3} = \dfrac{-6+3}{3} = \dfrac{-3}{3} = -1$

Equation of a circle having center (−3, −1) and r = 2 is

$(x-h)^2 + (y-k)^2 = r^2$

$(x+3)^2 + (h+1)^2 = 4$

When $h = -\dfrac{15}{13}$, $k = \dfrac{3}{13}$ then

Equation of a circle having center $\left(\dfrac{-15}{13}, \dfrac{3}{13}\right)$ and r = 2 is

$(x-h)^2 + (y-k)^2 = r^2$

$\left(x + \dfrac{15}{13}\right)^2 + \left(y - \dfrac{3}{13}\right)^2 = 4$

Q.3 Show that the circles $x^2 + y^2 + 2x - 2y - 7 = 0$ and $x^2 + y^2 - 6x - 4y + 9 = 0$ touch externally.

(Multan Board 2016 G-II, 2012 G-I)
(D.G.K Board 2017 G-I)(Lahore Board 2019 G-II)

Sol:

Equation of a given circle is

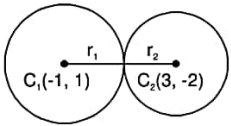
$x^2 + y^2 + 2x - 2y - 7 = 0$

Equation of a circle in general form is

$x^2 + y^2 + 2gx + 2fy + c = 0$

Here $2g = 2 \Rightarrow g = 1, 2f = -2 \Rightarrow f = -1 \Rightarrow c = -7$

Center $(-g, -f) = C_1(-1, 1)$



$r_1 = \sqrt{g^2 + f^2 - c} = \sqrt{1 + 1 + 7} = \sqrt{9} = 3$

And

Equation of a given circle is

$x^2 + y^2 - 6x + 4y + 9 = 0$

Equation of a circle in general form is

$x^2 + y^2 + 2gx + 2fy + c = 0$

Here $2g = -6 \Rightarrow g = -3,\ 2f = 4 \Rightarrow f = 2 \Rightarrow c = 9$

Center $(-g, -f) = C_2(3, -2)$

$r_2 = \sqrt{g^2 + f^2 - c} = \sqrt{9 + 4 - 9} = \sqrt{4} = 2$

$\overline{|C_1C_2|} = \sqrt{(3+1)^2 + (-2-1)^2}$

$= \sqrt{16 + 9} = \sqrt{25} = 5$

Since $\overline{|C_1C_2|} = r_1 + r_2$

This shows that given circles touch externally.

Q.4 Find equation of the circle of radius 2 and tangent to the line $x - y - 4 = 0$ at A $(1, -3)$

(Bahawalpur Board 2019)

Sol: Let C(h, k) be the center of the circle of radius 2 and tangent to the line $x - y - 4 = 0$ at A$(1, -3)$ then

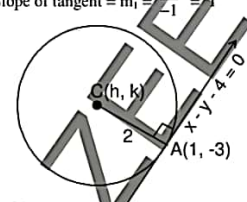
$|\overline{AC}|^2 = r^2$

$(h-1)^2 + (k+3)^2 = (2)^2$

$h^2 + 1 - 2h + k^2 + 9 + 6k = 4$

$h^2 + k^2 - 2h + 6k + 6 = 0 \quad ---(1)$

Slope of tangent $= m_1 = \frac{-1}{-1} = 1$



Slope of line $\overline{AC} = m_2 = \frac{k+3}{h-1}$

$\therefore \quad m_1 \times m_2 = -1 \Rightarrow 1 \times \frac{k+3}{h-1} = -1$

$k + 3 = -h + 1 \Rightarrow k = -h - 2 \quad ---(2)$

Put $k = -h - 2$ in eq. (1)

$h^2 + (-h-2)^2 - 2h + 6(-h-2) + 6 = 0$

$h^2 + h^2 + 4 + 4h - 2h - 6h - 12 + 6 = 0$

$2h^2 - 4h - 2 = 0 \Rightarrow h^2 - 2h - 1 = 0$

$h = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

Put $h = 1 + \sqrt{2}$ in eq. (2)

$k = -1 - \sqrt{2} - 2 = -3 - \sqrt{2}$

When $h = 1 + \sqrt{2}$, $k = -3 - \sqrt{2}$ then

Equation of a circle is

$(x-h)^2 + (y-k)^2 = r^2$

$(x - 1 - \sqrt{2})^2 + (y + 3 + \sqrt{2})^2 = 4$

Put $h = 1 - \sqrt{2}$ in eq. (2)

$k = -1 + \sqrt{2} - 2 = -3 + \sqrt{2}$

When $h = 1 - \sqrt{2}$, $k = -3 + \sqrt{2}$ then

Equation of a circle is

$(x-h)^2 + (y-k)^2 = r^2$

$(x - 1 + \sqrt{2})^2 + (y + 3 - \sqrt{2})^2 = 4$

MULTIPLE CHOICE QUESTIONS

☐ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 If a plane passes through the vertex of the cone being called. *(Faisalabad Board 2013)*

(a) Parabola (b) Hyperbola

(c) Unit circle (d) Point circle

Q.2 The length of the diameter of the circle $x^2 + y^2 = a^2$ is: *(D.G.K Board 2010, Faisalabad Board 2013)*

(a) a (b) 2a

(c) 1 (d) 2

Q.3 The circle $x^2 + y^2 + 2gx + 2fy + c = o$ pass through origin if: *(Rawalpindi Board 2013)*

(a) $c = o$ (b) $c = 1$

(c) -1 (d) $f + g$

Q.4 Radius of circle $4x^2 + 4y^2 + 8x + 8y - 6 = 0$ is. *(Sargodha Board 2013)*

(a) $4\sqrt{5}$ (b) $\sqrt{19}$

(c) $\sqrt{\frac{7}{2}}$ (d) 12

Q.5 Centre of circle $x^2 + y^2 + 7x - 3y = 0$ *(Sahiwal Board 2013)*

(a) $\left(\frac{-7}{2}, \frac{3}{2}\right)$ (b) $\left(\frac{7}{2}, \frac{-3}{2}\right)$

(c) (7, 3) (d) (−7, 3)

Q.6 Centre of the circle $5x^2 + 5y^2 + 14x + 12y - 10 = 0$ is: *(Multan Board 2013 G-II)* *(Faisalabad Board 2016)*

(a) $\left(\frac{-7}{5}, \frac{-6}{5}\right)$ (b) $\left(\frac{7}{6}, \frac{6}{5}\right)$

(c) (7, 6) (d) (7, −6)

Q.7 Centre of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is:
(Rawalpindi Board 2014)(Lahore Board 2018 G-I)
(a) (g, f) (b) (f, g)
(c) (−f, −g) (d) (−g, −f)

Q.8 The centre of the circle $x^2 + y^2 - 6x + 4y + 13 = 0$ is:
(Bahawalpur Board 2014)
(Rawalpindi Board 2017 G-I)(Sargodha Board 2013)
(Multan Board 2018 G-II)
(a) (−6, 4) (b) (6, −4)
(c) (3, −2) (d) (−3, 2)

Q.9 $x^2 + y^2 + 2gx + 2fy + c = 0$ is the equation of:
(D.G.K Board 2011-12 G-II, Multan Board 2014 G-I)
(a) Line (b) Ellipse
(c) Hyperbola (d) Circle

Q.10 ____ is equation of point circle:
(Multan Board 2014 G-II)
(a) $x^2 - y^2 = 7$ (b) $x^2 + y^2 = 4$
(c) $x^2 + y^2 = 0$ (d) $x^2 + y^2 = 1$

Q.11 A point circle is called a point circle if:
(D.G.K Board 2015 G-I, D.G.K Board 2014 G-II)
(a) r = 1 (b) r = 0
(c) r = 2 (d) r = 3

Q.12 General equation of circle is:
(D.G.K Board 2013 G-II)
(a) $x^2 + y^2 + 2gx + 2fy + C = 0$
(b) $x^2 + 2y^2 + 2gx + 2fy + C = 0$
(c) $2x^2 + y^2 + 2gx + 2fy + C = 0$
(d) $x^2 - y^2 + 2gx + 2fy + C = 0$

Q.13 Centre of point circle can be:
(a) (1, 1) (b) (0, 0)
(c) (1, 0) (d) All

Q.14 If the ends of the diameter of the circle are (0, 1) and (2, 3), then its area is.
(Lahore Board 2017 G-I)
(a) π (b) 2π
(c) 4π (d) 8π

Q.15 The radius of circle $x^2 + y^2 + 2gx + 2fg + C = 0$
(D.G.K Board 2010, Lahore Board 2014 G-II)
(Rawalpindi Board 2017)
(a) $\sqrt{g^2 + f^2 + C}$ (b) $\sqrt{g^2 - f^2 + C}$
(c) $\sqrt{g^2 + f^2 - C}$ (d) $\sqrt{g^2 - f^2 - C}$

Q.16 Radius of circle $x^2 + y^2 + 6x + 8y + 9 = 0$ is equal to: *(Sahiwal Board 2015)*
(a) 6 (b) 8
(c) 9 (d) 4

Q.17 Equation of circle with centre at origin and radius $\sqrt{5}$ is:
(Multan Board 2015 G-II)
(Sahiwal Board 2017)
(a) $x^2 + y^2 = \sqrt{5}$ (b) $x^2 + y^2 = 5$
(c) $x^2 + y^2 = 25$ (d) $x^2 - y^2 = 5$

Q.18 The centre of circle $5x^2 + 5y^2 - 10x + 5y + 1 = 0$ equals:
(Multan Board 2011 G-II)
(a) $\left(\frac{1}{2}, -1\right)$ (b) $\left(1, \frac{-1}{2}\right)$
(c) $\left(-1, \frac{1}{2}\right)$ (d) $\left(-1, \frac{-1}{2}\right)$

Q.19 The radius of circle $x^2 + y^2 - 6x + 8y = 0$ equals:
(Multan Board 2011 G-I)
(a) 13 (b) 12
(c) 4 (d) 5

Q.20 The set of all points in the plane that are equally distant from a fixed point is called.
(Lahore Board 2015 G-I, 2018 G-II)
(a) Ellipse (b) Parabola
(c) Hyperbola (d) Circle

Q.21 Centre of circle $x^2 + y^2 + 4x + 6y + 3 = 0$.
(Lahore Board 2012 G-II)
(a) (2, 3) (b) (−2, 3)
(c) (−2, −3) (d) (2, −3)

Q.22 Center of circle having equation. $x^2 + y^2 - 12x - 10y = 0$ is.
(D.G.K Board 2017 G-I)
(a) (6, −5) (b) (−6, 5)
(c) (−6, −5) (d) (6, 5)

Q.23 The equation $Ax^2 + By^2 + Gx + Fy + c = 0$ represents a circle if. *(Gujranwala Board 2016)*
(a) A = 0, B ≠ 0 (b) A ≠ B
(c) A = B ≠ 0 (d) C = 0

Q.24 A circle touches the two axis at (a, 0) and (0, a), then centre of circle is:
(Rawalpindi Board 2016)
(a) (−a, a) (b) (a, −a)
(c) (a, a) (d) (−a, −a)

Q.25 If a point lies inside a circle, then its distance from the centre is:
(a) Equal to the radius
(b) Less than the radius
(c) Greater than the radius
(d) Equal to or greater than the radius

Q.26 The curves obtained by cutting a double right circular cone by a ---------- are called conics.
(a) Straight line (b) Plane
(c) Curve (d) None of these

Q.27 The two parts of a right circular cones are called.
(a) Nappes (b) Apex of the cone
(c) Generator (d) Vertex

Q.28 The fixed point on the conic is called.
(a) Directrix (b) Vertex
(c) Focus (d) None of these

Q.29 If the cone is cut by a plane perpendicular to the axis of the cone, then the section is a\an.
(a) Parabola (b) Circular cone
(c) Ellipse (d) Circle

Q.30 If the cutting plane is slightly tilled and cuts only one nappe of the cone, then the section is a\an.
(a) Ellipse (b) Circular cone
(c) Circle (d) Point circle

Q.31 If the cutting plane is parallel to the axis of the cone and intersects both of its nappes, then the section is a\an.
(a) Parabola (b) Hyperbola
(c) Ellipse (d) None of these

Q.32 If equation of circle is $(x - h)^2 + (y - k)^2 = r^2$, then centre of a circle is:
(a) $(-h, -k)$ (b) (h, k)
(c) $(-h, k)$ (d) $(h, -k)$

Q.33 The centre of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is:
(a) $(-g, -f)$ (b) $(-f, -g)$
(c) $(0, 0)$ (d) None of these

Q.34 If r is the radius of the circle and its centre is at origin, then equation of circle is:
(a) $x^2 + y^2 = a^2$ (b) $x^2 + y^2 = r^2$
(c) $x^2 - y^2 = a^2$ (d) $x^2 - y^2 = r^2$

Q.35 If (h, k) and r is the centre and radius of the circle respectively, then equation of a circle in standard form is:
(a) $(x + h)^2 + (y + k)^2 = r^2$
(b) $(x - h)^2 + (y - k)^2 = r^2$
(c) $(x - h)^2 - (x - k)^2 = r^2$
(d) $x^2 + y^2 = r^2$

Q.36 In equation of circle, coefficient of each of x^2 and y^2 are:
(a) Not equal (b) Opposite in signs
(c) Equal (d) None of these

Q.37 A line segment whose end points lie on the circle is called a ---------- of the circle.
(a) Radius (b) Chord
(c) Diameter (d) None of these

Q.38 A line touching a circle is called.
(a) Tangent (b) Secant
(c) Chord (d) Radial chord

Q.39 A fixed line is called a ---------- of the conic.
(a) Vertex (b) Directrix
(c) Focus (d) None of these

Q.40 A point circle has:
(a) Any point on it (b) (0, 0)
(c) Centre only (d) None of these

Q.41 A line segment having both the end-points on a circle and passing through the centre of a circle is known as
(a) Diameter (b) Secant
(c) Radial chord (d) Chord

Q.42 Two circles of radius 3 cm and 4 cm touch each other externally. The distance between their centres is:
(a) 1 cm (b) 4 cm
(c) 5 cm (d) 7 cm

Q.43 The distance from the centre of the circle to any point on the circle is called the.
(a) Radius of the circle
(b) Diameter of the circle
(c) Centre of the circle
(d) None of these

Q.44 If the cutting plane is parallel to the generator of the cone, but intersects its one nappe only, then section is a\an.
(a) Parabola (b) Ellipse
(c) Hyperbola (d) None of these

Q.45 A chord containing the centre of the circle is called ---------- of the circle.
(a) Diameter (b) Chord
(c) Radius (d) None of these

Q.46 Centre of circle $4x^2 + 4y^2 - 8x + 16y - 25 = 0$. *(Lahore Board 2013 G-I)*
(a) $\left(1, -\frac{3}{2}\right)$ (b) $\left(-\frac{3}{2}, 1\right)$
(c) $(1, -2)$ (d) $(1, 2)$

Q.47 Center of circle $(x - 1)^2 + (y + 3)^2 = 3$ is. *(A.J.K Board 2017)*
(a) $(-1, -3)$ (b) $(-1, 3)$
(c) $(-1, -3)$ (d) $(1, 3)$

Q.48 Center of circle $x^2 + y^2 = r^2$ is. *(Bahawalpur Board 2018)*
(a) $(1, 1)$ (b) $(2, 0)$
(c) $(0, 0)$ (d) $(0, 2)$

Q.49 The length of diameter of circle $(x + 5)^2 + (y - 8)^2 = 12$ is. *(Sahiwal Board 2018)*
(a) $4\sqrt{3}$ (b) $2\sqrt{3}$
(c) 12 (d) 24

Q.50 The length of diameter of the circle $(x + 8)^2 + (y - 5)^2 = 80$ is. *(Multan Board 2018 G-I)*
(a) 160 (b) $4\sqrt{5}$
(c) $8\sqrt{5}$ (d) 40

Q.51 The center of circle $(x + 3)^2 + y - 2)^2 = 16$ equals. *(Rawalpindi Board 2018)*
(a) $(3, -2)$ (b) $(-3, 2)$
(c) $(3, 2)$ (d) $(-3, -2)$

Q.52 Length of diameter of circle $(x-5)^2+(y-2)^2=8$ is. *(Sargodha Board 2018)*

(a) 64 (b) 16

(c) $2\sqrt{2}$ (d) $4\sqrt{2}$

Q.53 The center of the circle $(x-1)^2+(y+3)^2=9$ is: *(Faisalabad Board 2019)*

(a) (–1 , 3) (b) (–1 , –3)

(c) (1 , 3) (d) (1 , –3)

Q.54 Two circles are said to be concentric circles if they have: *(Faisalabad Board 2019 G-II)*

(a) Same radius (b) Different center

(c) Same center (d) Same diameter

Q.55 Centre of the circle $5x^2+5y^2+14x+12y-10=0$ is *(Gujranwala Board 2019 G-I)*

(a) $\left(\frac{-7}{5},\frac{-6}{5}\right)$ (b) $\left(\frac{7}{5},\frac{6}{5}\right)$

(c) (7 , 6) (d) (7, -6)

Q.56 The centre of the circle $x^2+y^2-6x+4y+13=0$ is *(Gujranwala Board 2019 G-II)*

(a) (3, 2) (b) (3, –2)

(c) (2, 3) (d) (–2, –3)

Q.57 The radius of circle $x^2+y^2=5$ is: *(Lahore Board 2019)*

(a) 25 (b) $\sqrt{5}$

(c) 5 (d) (0, 0)

Q.58 Radius of circle $x^2+y^2-4x-6y=0$ is: *(Multan Board 2019)*

(a) $\sqrt{13}$ (b) $\sqrt{11}$

(c) $\sqrt{5}$ (d) 13

Q.59 The centre of circle $(x+1)^2+(y-2)^2=26$ is: *(Rawalpindi Board 2019)*

(a) (1 , 2) (b) (–1 , 2)

(c) (–1 , –2) (d) (1 , –2)

Q.60 Eccentricity e of circle is *(Sahiwal Board 2019)*

(a) $e<1$ (b) $e=1$

(c) $e>1$ (d) $e=0$

Q.61 Radius of circle $x^2+y^2+2gx+2fy-c=0$ is *(Sahiwal Board 2019)*

(a) $\sqrt{g^2+f^2-c}$ (b) $\sqrt{g^2+f^2+c}$

(c) $\sqrt{g^2+f^2-c^2}$ (d) $\sqrt{g^2+f^2+c^2}$

EXERCISE 6.2

SHORT ANSWERS TO THE QUESTIONS

Q.1 Define a tangent line and normal line of a curve.

Ans. A tangent to a curve is a line that touches the curve cutting through it. The normal to the curve at P is the line through p perpendicular to the tangent to the curve at p.

Q.2 Write down an equation of tangent to the circle $x^2+y^2=25$ at (4, 3). *(Gujranwala Board 2011) (Rawalpindi Board 2013, 2014) (Lahore Board 2010, 2014 G-II, 2011, 2014 G-I)*

Ans. Equation of tangent to circle

$x^2+y^2=25$ at (4, 3) is

$xx_1+yy_1=25$

Put, $P(x_1, y_1)=P(4, 3)$

$4x+3y=25$

Required equation.

Q.3 Write down equation of the tangent to the circle $x^2+y^2=25$ at (5 cos θ, 5 sin θ) *(Gujranwala Board 2010)*

Ans. Given that $x^2+y^2=25$ ……(i)

Differentiating (i) w.r.t. 'x', we have

$$2x+2y\frac{dy}{dx}=0 \Rightarrow y\frac{dy}{dx}=-x \Rightarrow \frac{dy}{dx}=\frac{-x}{y}$$

$$\left(\frac{dy}{dx}\right)(5\cos\theta, 5\sin\theta)=\frac{-5\cos\theta}{5\sin\theta}=-\frac{\cos\theta}{\sin\theta}$$

Equation of tangent to the circle (i) at (5 cos θ, 5 sin θ) is

$$y-5\sin\theta=-\frac{\cos\theta}{\sin\theta}(x-5\cos\theta)$$

$y\sin\theta-5\sin^2\theta=-x\cos\theta+5\cos^2\theta$

$x\cos\theta+y\sin\theta=5\sin^2\theta+5\cos^2\theta$

$x\cos\theta+y\sin\theta=5(1)$

$x\cos\theta+y\sin\theta=5$

Q.4 Find the length of the tangent from the point P(-5, 10) to the circle $5x^2+5y^2+14x+12y-10=0$ *(Bahawalpur Board 2019) (Multan Board 2013 G-I)(Sargodha Board 2016)*

Ans. Solution:

$5x^2+5y^2+14x+12y-10=0$

Divide by 5

$$x^2+y^2+\frac{14}{5}x+\frac{12}{5}y-2=0$$

Length of the tangent $=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$

$$=\sqrt{(5)^2+(10)^2+\frac{14}{5}(-5)+\frac{12}{5}(10)-2}$$

$$=\sqrt{25+100+(14)(-1)+12(2)-2}$$

$$=\sqrt{125-14+24-2}=\sqrt{133}$$

Q.5 Find the length of the tangent drawn from the point (–5, 4) to the circle

$$5x^2 + 5y^2 - 10x + 15y - 131 = 0$$

(Lahore Board 2012 G-I, 2015 G-II)
(Multan Board 2014 G-II) (Rawalpindi Board 2016)
(Gujranwala Board 2019 G-II)

Ans. Standard form of the given equation of circle is

$$x^2 + y^2 - 2x + 3y - \frac{131}{5} = 0$$

The length of tangent drawn from (–5, 4) to the circle

$$= \sqrt{(-5)2 + 42 - 2(-5) + 3(4) - \frac{131}{5}}$$

$$= \sqrt{25 + 16 + 10 + 12 - \frac{113}{5}} = \sqrt{63 - \frac{131}{5}}$$

$$= \sqrt{\frac{315 - 131}{5}} = \sqrt{\frac{184}{5}}$$

Q.6 Find equation of the tangent to the circle $x^2 + y^2 = 2$ perpendicular to the line $3x + 2y = 6$.

(Bahawalpur Board 2016)

Ans. The slope of the given line $= -\frac{3}{2}$

As the tangent is perpendicular to the given line, so its slope

$$= m = -\left(\frac{-2}{3}\right) = \frac{2}{3}.$$

In this case radius $= a = \sqrt{2}$. The equations of the tangents to the circle $x^2 + y^2 = 2$ having slope $\frac{2}{3}$ are

$$y = \frac{2}{3}x \pm \sqrt{2}\sqrt{1 + \frac{4}{9}}$$

$\left(\text{Using } y = mx \pm a\sqrt{1 + m^2}\right)$

$$y = \frac{2}{3}x \pm \sqrt{2}\sqrt{\frac{9 + 4}{9}}$$

$$y = \frac{2}{3}x \pm \sqrt{2}\left(\frac{\sqrt{13}}{3}\right)$$

$$y = \frac{2}{3}x \pm \frac{\sqrt{26}}{3}$$

$$3y = 2x \pm \sqrt{26}$$

Thus the equations of tangents are

$$2x - 3y + \sqrt{26} = 0 \text{ and } 2x - 3y - \sqrt{26} = 0$$

LONG QUESTIONS

Q.1 Find the length of chord cut off from the line $2x + 3y = 13$ by the circle $x^2 + y^2 = 26$.

(Rawalpindi Board 2017 G-II)
(D.G.K Board 2014, 2017 G-II)
(Lahore Board 2016 G-I)

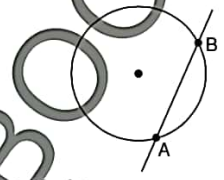
Sol: Equation of the circle is

$$x^2 + y^2 = 26 \quad ---(1)$$

Equation of the line is

$$2x + 3y = 13$$

$$2x = 13 - 3y \Rightarrow x = \frac{13 - 3y}{2} \quad ---(2)$$



Put $x = \frac{13 - 3y}{2}$ in equation (1)

$$\left(\frac{13 - 3y}{2}\right)^2 + y^2 = 26$$

$$\frac{169 + 9y^2 - 78y}{4} + y^2 - 26 = 0$$

$$\frac{169 + 9y^2 - 78y + 4y^2 - 104}{4} = 0$$

$$13y^2 - 78y + 65 = 0 \Rightarrow y^2 - 6y + 5 = 0$$

$$(y - 5)(y - 1) = 0$$

$$y = 5, y = 1$$

Put $y = 5$ in equation (2)

$$x = \frac{13 - 3(5)}{2} = \frac{13 - 15}{2} = \frac{-2}{2} = -1$$

$\therefore$ A(–1, 5)

Put $y = 1$ in equation (2)

$$x = \frac{13 - 3(1)}{2} = \frac{13 - 3}{2} = \frac{10}{2} = 5$$

$\therefore$ B(5, 1)

$$\text{Length of chord} = |\overline{AB}| = \sqrt{(5 + 1)^2 + (1 - 5)^2}$$

$$= \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$$

Q.2 Find equations of the tangents to the circle $x^2 + y^2 = 2$ perpendicular to the line $3x + 2y = 6$.

(Lahore Board 2013)(Faisalabad Board 2016 G-I)
(Multan Board 2018 G-I)

Sol: See Short Question 17

Q.3 Find the equation o tangents draw from the point $(-7, -2)$ to the circle $(x + 1)^2 + (y - 2)^2 = 26$.

(Sahiwal Board 2014)

Sol: Equation of a circle is

$$(x + 1)^2 + (y - 2)^2 = 26 \quad ---(A)$$

$$(x + 1)^2 + (y - 2)^2 = (\sqrt{26})^2$$

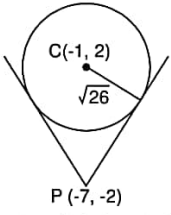
$C(-1, 2)$ and $r = \sqrt{26}$

Let equation of tangent from $P(-7, -2)$ is

$$y - y_1 = m(x - x_1)$$

$$y + 2 = m(x + 7) \Rightarrow y + 2 = mx + 7m$$

$$mx - y + 7m - 2 = 0 \quad ---(1)$$



If equation (1) is tangent line to circle then its distance from $C(-1, 2)$ is equal to the radius of the circle

$$r = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$\sqrt{26} = \frac{|m(-1) - 1(2) + 7m - 2|}{\sqrt{m^2 + 1}}$$

$$\Rightarrow \sqrt{26m^2 + 26} = |-m - 2 + 7m - 2|$$

$$\sqrt{26m^2 + 26} = |6m - 4|$$

Squaring both sides

$$26m^2 + 26 = 36m^2 + 16 - 48m$$

$$10m^2 - 48m - 10 = 0 \Rightarrow 5m^2 - 24m - 5 = 0$$

$$5m^2 - 25m + 1m - 5 = 0$$

$$5m(m - 5) + 1(m - 5) = 0 \Rightarrow (m - 5)(5m + 1) = 0$$

$$m = 5, \ m = \frac{-1}{5}$$

Put $m = \frac{-1}{5}$ in eq (1)

$$\left(\frac{-1}{5}\right)x - y + 7\left(\frac{-1}{5}\right) - 2 = 0$$

$$-x - 5y - 7 - 10 = 0$$

$$\Rightarrow x + 5y + 17 = 0$$

Put $m = 5$ in eq (1)

$$5x - y + 7(5) - 2 = 0$$

$$5x - y + 33 = 0$$

Now find the points of contact of

$$5x - y + 33 = 0 \text{ and } (x + 1)^2 + (y - 2)^2 = 26$$

Equation of the circle is

$$(x + 1)^2 + (y - 2)^2 = 26 \quad ---(A)$$

Equation of the line is

$$5x - y + 33 = 0 \Rightarrow y = 5x + 33 \quad ---(1)$$

Put $y = 5x + 33$ in equation (A)

$$(x + 1)^2 + (5x + 33 - 2)^2 = 26$$

$$(x + 1)^2 + (5x + 31)^2 = 26$$

$$x^2 + 1 + 2x + 25x^2 + 961 + 310x - 26 = 0$$

$$26x^2 + 312x + 936 = 0$$

$$\Rightarrow \quad 13x^2 + 156x + 468 = 0$$

$$x = \frac{-156 \pm \sqrt{(156)^2 - 4(13)(468)}}{2(13)}$$

$$x = \frac{-156 \pm \sqrt{24336 - 24336}}{26} = \frac{-156 \pm \sqrt{0}}{26}$$

$$x = \frac{-156}{26} = -6$$

Put $x = -6$ in equation (1)

$$y = 5(-6) + 33 = 3$$

$\therefore$ $A(-6, 3)$

Now find the points of contact of

$$x + 5y + 17 = 0 \text{ and } (x + 1)^2 + (y - 2)^2 = 26$$

Equation of the circle is

$$(x + 1)^2 + (y - 2)^2 = 26 \quad ---(A)$$

Equation of the line is

$$x + 5y + 17 = 0 \Rightarrow x = -5y - 17 \quad ---(2)$$

Put $x = -5y - 17$ in equation (A)

$$(-5y - 17 + 1)^2 + (y - 2)^2 = 26$$

$$(-5y - 16)^2 + (y - 2)^2 = 26$$

$$25y^2 + 256 + 160y + y^2 + 4 - 4y - 26 = 0$$

$$26y^2 + 156y + 234 = 0$$

$$\Rightarrow \quad 13y^2 + 78y + 117 = 0$$

$$y = \frac{-78 \pm \sqrt{(78)^2 - 4(13)(117)}}{2(13)}$$

$$y = \frac{-78 \pm \sqrt{6084 - 6084}}{26} = \frac{-78 \pm \sqrt{0}}{26}$$

$$y = \frac{-78}{26} = -3$$

Put $y = -3$ in equation (2)

$$x = -5(-3) - 17 = 15 - 17 = -2$$

$\therefore$ $B(-2, -3)$

Q.4 Find a joint equation to the pair of tangent drawn from (5, 0) to the circle $x^2 + y^2 = 9$

(Lahore Board 2017 G-II)

Sol: equation of a circle is:

$x^2 + y^2 = 9 \quad \Rightarrow \quad x^2 + y^2 = (3)^2$

$\therefore \quad r = 3$ and $C(0, 0)$

The equation of the tangent drawn from A(5, 0) is:

$y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 0 = m(x - 5)$

$y = m(x - 5)$

$y = mx - 5m$

$\Rightarrow \quad mx - y - 5m = 0 \quad \ldots\ldots (1)$

If equation (1) is tangent line to circle then its distance from C(0, 0) is equal to the radius of the circle

Now, r = Distance of C(0, 0) from eq. (1)

$$3 = \frac{|(m)(0) + (-1)(0) - 5m|}{\sqrt{m^2 + (-1)^2}}$$

$$= \frac{|0 - 0 - 5m|}{\sqrt{m^2 + 1}} = \frac{|-5m|}{\sqrt{m^2 + 1}}$$

Squaring both the sides:

$$(3)^2 = \frac{|-5m|^2}{m^2 + 1} \quad \Rightarrow \quad 9 = \frac{25m^2}{m^2 + 1}$$

$9(m^2 + 1) = 25m^2 \quad \Rightarrow \quad 9m^2 + 9 = 25m^2$

$25m^2 - 9m^2 - 9 = 0 \quad \Rightarrow \quad 16m^2 - 9 = 0$

$16m^2 = 9 \Rightarrow m^2 = \frac{9}{16}$

$\Rightarrow \quad m = \sqrt{\frac{9}{16}} \Rightarrow m = \pm\frac{3}{4}$

When $m = \frac{3}{4}$, then from eq. (1)

$$\frac{3}{4}x - y - 5\left(\frac{3}{4}\right) = 0 \Rightarrow 3x - 4y - 15 = 0$$

When $m = -\frac{3}{4}$, then from eq. (1)

$$-\frac{3}{4}x - y - 5\left(-\frac{3}{4}\right) = 0$$

$\Rightarrow \quad -3x - 4y + 15 = 0 \Rightarrow \quad 3x + 4y - 15 = 0$

For joint equation

$\therefore \quad (3x - 4y - 15)(3x - 4y - 15) = 0$

$[3(x - 5) - 4y][3(x - 5) + 4y]$

$= 0 \Rightarrow (3(x - 5))^2 - (4y)^2 = 0$

$9(x - 5)^2 - 16y^2 = 0$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 The equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy = 0$ at the origin is: *(D.G.K Board 2015 G-II)*

(a) $x = 0$ (b) $y = 0$

(c) $fx + gy = 0$ (d) $gx + fy = 0$

Q.2 The length of tangent from (1, 2) to circle $x^2 + y^2 + 2x + 2 = 0$ is: *(D.G.K Board 2011)*

(a) 1 (b) 2

(c) 0 (d) 3

Q.3 The condition that the line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ is: *(Gujranwala Board 2014)*

(a) $c^2 = a^2(1 + m^2)$ (b) $c^2 = a^2(1 - m^2)$

(c) $c^2 = a^2(m^2 - 1)$ (d) $c = a^2(1 + m^2)$

Q.4 Length of tangent from (0, 1) to $x^2 + y^2 + 6x - 3y + 3 = 0$. *(Lahore Board 2015 GI-II)*

(a) 2 (b) 3

(c) 4 (d) 1

Q.5 The lenfgth of tangent drawn from the point (1, 1) to the circle $x^2 + y^2 - 3x + 9y + 8 = 0$ is: *(Gujranwala 2013)*

(a) 1 (b) 8

(c) 4 (d) 16

Q.6 The point (x_1, y_1) lies on the circle $x^2 + y^2 + 2gx + fy + c = 0$.

(a) $x_1^2 + y_1^2 + 2g x_1 + 2fy_1 + c = 0$

(b) $x_1^2 + y_1^2 + 2g x_1 + 2fy_1 + c > 0$

(c) $x_1^2 + y_1^2 + 2g x_1 + 2fy_1 + c < 0$

(d) None of these

Q.7 If r is the radius of any circle and c its centre, then any point $P(x_1, y_1)$ lies outside the circle only if.

(a) $|CP| < r$ (b) $|CP| = r$

(c) $|CP| > r$ (d) None of these

Q.8 The point (x_1, y_1) lies outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then

(a) $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$

(b) $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$

(c) $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

(d) None of these

Q.9 If r is the radius of any circle and c its centre, then any point $P(x_1, y_1)$ lies on the circle only if:

(a) $|CP| < r$ (b) $|CP| > c$

(c) $|CP| = r$ (d) None of these

Q.10 Two coincident tangents can be drawn to a circle from any point $P(x_1, y_1)$ ---------- the circle.

(a) Inside (b) On

(c) Outside (d) None of these

Q.11 A line through a point say P perpendicular to the tangent to the curve at P is called.

(a) Straight line (b) Tangent line

(c) Normal line (d) None of these

Q.12 The point (x_1, y_1) lies inside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

(a) $x_1^2 + y_1^2 + 2g x_1 + 2fy_1 + c = 0$

(b) $x_1^2 + y_1^2 + 2g x_1 + 2fy_1 + c > 0$

(c) $x_1^2 + y_1^2 + 2g x_1 + 2fy_1 + c < 0$

(d) None of these

Q.13 Any line $y = mx + c$ intersects any circle $x^2 + y^2 = r^2$ in

(a) At least two points

(b) two points

(c) At most two points

(d) One point

Q.14 Condition that the line $y = mx + c$ is tangent to the circle $x^2 + y^2 = a^2$ is. *(Faisalabad Board 2017)*

(a) $c = \pm m\sqrt{1 + a^2}$ (b) $c = \pm a\sqrt{1 + m^2}$

(c) $c = \pm a\sqrt{1 - m^2}$ (d) $c = \pm m\sqrt{1 - a^2}$

Q.15 Equation of tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is. *(Multan Board 2017 G-I)*

(a) $\sqrt{3}\,x + y = 4$ (b) $\sqrt{3}\,x - y = 4$

(c) $x - \sqrt{3}\,y = 4$ (d) $x + \sqrt{3}\,y = 4$

Q.16 If r is the radius of any circle and c its centre, then any point $P(x_1, y_1)$ lies inside the circle only if

(a) $|CP| < r$ (b) $|CP| = r$

(c) $|CP| > r$ (d) None of these

Q.17 The line $y = mx + c$ will be tangent to the circle $x^2 + y^2 = a^2$ if *(Sargodha Board 2018)*

(a) $c = \frac{a}{m}$ (b) $c = \pm a\sqrt{1 - m^2}$

(c) $c = \pm a\sqrt{1 + m^2}$ (d) $c = \pm a\sqrt{m^2 - 1}$

Q.18 The length of diameter of the circle $x^2 + y^2 - 4x - 12 = 0$ is: *(Bahawalpur Board 2019)*

(a) 6 (b) 7

(c) 8 (d) 9

Q.19 The length of tangent from (0 , 1) to the circle $x^2 + y^2 + 6x - 3y + 3 = 0$ is *(Gujranwala Board 2019 G-I)*

(a) 2 (b) 3

(c) 4 (d) 1

EXERCISE 6.3

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 Perpendicular dropped from the centre of a circle on a chord -------- the chord.

(a) Normal (b) Bisects

(c) Equal to (d) None of these

Q.2 Measure of the central angle of a minor arc is ---------- the measure of the angle subtended in the corresponding major arc.

(a) Equal (b) Double

(c) Not equal to (d) None of these

Q.3 If a circle and a line intersect in two points, then the line is called.

(a) A chord (b) A secant

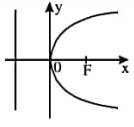
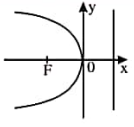
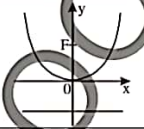
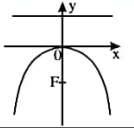
(c) A diameter (d) None of the above

Q.4 An angle in a semi-circle is:

(a) 0° (b) 90°

(c) 180° (d) None of these

SUMMARY OF STANDARD PARABOLAS

Sr.No.	1	2	3	4
Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Focus	(a, 0)	(−a, 0)	(0, a)	(0, −a)
Directrix	x = −a	x = a	y = −a	y = a
Vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Axis	y = 0	y = 0	x = 0	x = 0
Latusrectum	x = a	x = −a	y = a	y = −a
Graph				

EXERCISE 6.4

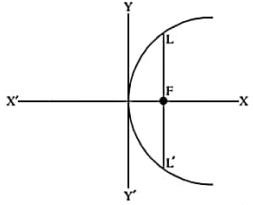
SHORT ANSWERS TO THE QUESTIONS

Q.1 Define conics in terms of eccentricity, focus, and directrix and define eccentricity of conic.
(Faisalabad Board 2019 G-II)

Ans. Let e is a fixed number. A conic with eccentricity e, focus F, and directrix L is the set of all points P in the plane such that the distance from P to F, divided by the distance from P to L, is equal to e.

Q.2 Define latusrectum of the parabola.
(Rawalpindi Board 2013)(Sahiwal Board 2019)

Ans. Latusrectum of the Parabola:
The chord of parabola passes through focus F(a, 0) and $\perp$ to axis of parabola is called latus rectum.
In figure LL′ is the Latus rectum.



Q.3 Find the focus and vertex of parabola $x^2 = -16y$.
(Lahore Board 2013)(Gujranwala Board 2014, 2016)
(Faisalabad Board 2016, 2019 G-II)
(Sargodha Board 2017)(Rawalpindi Board 2019)
(D.G Khan Board 2017 G-II)

Ans. $x^2 = -16y$ ..... (1)
Comparing (i) with $x^2 = 4ay$
$4a = 16$
$a = 4$
Focus is $F(0, -a) = F(0, -4)$
Vertex is at origin i.e.
Vertex $= V(0,0)$

Q.4 Find the focus of the parabola
$x + 8 - y^2 + 2y = 0$ *(Gujranwala Board 2018)*

Ans. Given equation of parabola is
$x + 8 - y^2 + 2y = 0$
or $y^2 - 2y = x + 8$ Adding 1 on both sides
$y^2 - 2y + 1 = x + 8 + 1$
$(y-1)^2 = x + 9$ ..... (1)
Shift the origin to (−9, 1), we have
$x = X - 9$, $y = Y + 1$ referred to XY-axes with (−9 ,1) as origin.
$x + 9 = X$, $y - 1 = Y$
The equation (1) becomes $Y^2 = X$ ..... (2)
Which is a parabola, whose focus lies on $Y = 0$, coordinates of the focus of (2) are

$$X = \frac{1}{4}, Y = 0 \Rightarrow x + 9 = \frac{1}{4}, y - 1 = 0$$

$$\Rightarrow \quad x = \frac{1}{4} - 9, \; y = 1 \Rightarrow x = -\frac{35}{4}, \; y = 1$$

Thus coordinates of the focus of the parabola of (1) are $\left(-\frac{35}{4}, 1\right)$

Q.5 Find focus, vertex and equation of directrix of the parabola $x^2 - 4x - 8y + 4 = 0$.
(Lahore Board 2011 G-I)(Gujranwala Board 2013)
(Multan Board 2018 G-I)

Ans. $x^2 - 4x - 8y + 4 = 0$
$x^2 - 4x + 4 - 8y = 0$
$(x-2)^2 - 8y = 0$
$(x-2)^2 = 8y$
Put, $x - 2 = X$ and $y = Y$
$X^2 = 8Y$ ..... (i)
Which is the equation of parabola comparing (i) with $X^2 = 4ay$
$4a = 8$ or $a = 2$
Focus is $F(0, a) = F(0, 2)$
$\Rightarrow \quad X = 0$, $Y = 2$
$x - 2 = 0$
$x = 2 = 0$, $y = 2$
$x = 2$
Focus is F(2, 2).
Equation of direction is;
$Y = -a$
$y = -2$
Vertex, $X = 0$, $Y = 0$
$x - 2 = 0$, $y = 0$
V(2, 0)

Q.6 Find the focus and vertex of the parabola
$x^2 = 4(y - 1)$ *(Sahiwal Board 2018)*
(Lahore Board 2014 G-II)(A.J.K Board 2017)
(Gujranwala Board 2019 G-II)

Ans. Given equation of parabola is
$x^2 = 4(y-1)$ ..... (1)
Shifting the origin to (0, 1), we have
$x = X + 0$ and $y = Y + 1$
referred to XY-axis with (0,1) as origin.
$x = X$, $y - 1 = Y$
Then equation (1) becomes $X^2 = 4Y$ .....(2)

Which is a parabola whose focus lies on X = 0, coordinates of the focus of (2) are X = 0, Y = 1 i.e., x = 0, y – 1 = 1 or x = 0, y = 2

Thus coordinates of the focus of the parabola (1) are (0, 2)

Vertex of (2) has coordinates X = 0, Y = 0

i.e., x = 0, y – 1 = 0

x = 0, y = 1 are the coordinates of the vertex of (1)

Q.7 Write an equation of the parabola with focus (–3, 1) and directrix x = 3.

(Sargodha Board 2013)(Multan Board 2014 G-I, II) (Gujranwala Board 2010)(Multan Board 2017 G-I) (Multan Board 2013)(Sargodha Board 2018)

Ans. Let P(x, y) be any point of parabola then

$$|PF| = |PM|$$

$$\sqrt{(x+3)^2 + (y-1)^2} = \frac{|x-3|}{\sqrt{(1)^2 + (0)^2}}$$

Squaring on both sides

$(x + 3)^2 + (y – 1)^2 = (x – 3)^2$

$x^2 + 6x + 9 + y^2 – 2y + 1 = x^2 + 9 – 6x$

$y^2 + 12x – 2y + 1 = 0$

$y^2 – 2y + 12x + 1 = 0$

Q.8 Find an equation of the parabola whose focus is F(–3, 4) and directrix is 3x – 4y + 5 = 0.

(Lahore Board 2013 G-II)

Ans. Let P(x, y) be a point on the parabola. Length of the perpendicular |PM| from P(x, y) to the directrix 3x – 4y + 5 = 0 is given by

$$|PM| = \frac{|3x - 4y + 5|}{\sqrt{3^2 + (-4)^2}}$$

By definition, |PF| = |PM|

or $|PF|^2 = |PM|^2$

or $(x + 3)^2 + (y – 4)^2 = \frac{(3x – 4y + 5)}{25}$

or $25(x^2 + 6x + 9 + y^2 – 8y + 16)$
$= 9x^2 + 16y^2 + 25 – 24xy + 30x – 40y$

or $16x^2 + 24xy + 9y^2 + 120x – 160y + 375 = 0$

is an equation of the required parabola.

Q.9 Find the equation of parabola with Directrix y = 1, length of latus-rectum is 8. Opens downward.

Ans. Here LL′ = 8 = 4a

As parabola opens downward, so equation of parabola is

$(x – h)^2 = –4a (y – k)$ ..... (1)

But k = 1, because distance between vertex and directrix is equal to distance between vertex and focus i.e. two and sign is negative because of downward opening.

Thus equation (1) becomes

$(x – h)^2 = – (8) (y + 1)$

$x^2 – 2xh + h^2 = –8y – 8$

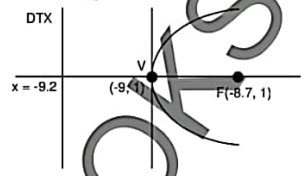
$x^2 – 2hx + 8y + 8 + h^2 = 0$

LONG QUESTIONS

Q.1 Find focus, vertex and directrix of

$x + 8 – y^2 + 2y = 0$ *(Sargodha Board 2017)*

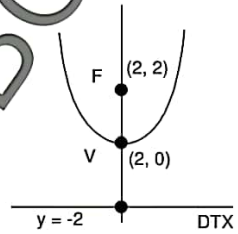
Sol: See Short Question 11



Q.2 Find focus, vertex and the equation of directrix of parabola $x^2 – 4x – 8y + 4 = 0$

(Lahore Board 2015 G-II)

Sol: See Short Question 12



Q.3 Find an equation of parabola with focus (–3, 1) and directrix x – 2y – 3 = 0.

(Multan Board 2012 G-I, 2015 G-II)

Sol: See Short Question 22

Q.4 Writ the equation of parabola whose focus (–3, 1), directrix x = 3. *(Multan Board 2016 G-II)*

Sol: See Short Question 19

Q.5 Find equation of the parabola having the focus at the origin and directrix parallel to y-axis.

(Sahiwal Board 2015)(Lahore Board 2018 G-II)

Sol: Let P(x, y) be any point on the parabola with the focus (0, 0) and directrix x = –2a or x + 2a = 0. Then:

$$\overline{|PM|} = \overline{|PF|} \Rightarrow \sqrt{(x-0)^2 + (y-0)^2} = \frac{|(1)(x) + (0)(y) + 2a|}{\sqrt{1^2 + 0^2}}$$

$$\sqrt{x^2 + y^2} = |x + 2a|$$

Squaring both sides:

$x^2 + y^2 = |x + 2a|^2 \quad \Rightarrow \quad x^2 + y^2 = x^2 + 4ax + 4a^2$

$y^2 = 4ax + 4a^2 \quad \Rightarrow \quad y^2 – 4ax – 4a^2 = 0$

Q.6 Show that an equation of the parabola with focus at (a cos α, a sin α) and directrix
x cos α + y sin x α + a = 0 is.
(Lahore Board 2013 G-II)
(Gujranwala Board 2011, 2019 G-II))

Sol: Let P (x, y) be a point on the parabola with the focus F (a cos α , a sin α) and directrix
x cos α + y sin α + a ≡ 0
Be definition
|PF| = |PM|

$$\sqrt{(x - a\cos\alpha)^2 + (y - a\sin\alpha)^2} = \frac{|x\cos\alpha + y\sin\alpha + a|}{\sqrt{(\cos\alpha)^2 + (\sin\alpha)^2}}$$

Squaring both sides
$x^2 + a^2\cos^2\alpha - 2ax\cos\alpha + y^2 + a^2\sin^2\alpha - 2ay\sin\alpha$
$= x^2\cos^2\alpha + y^2\sin^2\alpha + a^2 + 2xy\cos\alpha\sin\alpha$
$+ 2ay\sin\alpha + 2ax\cos\alpha$
$x^2(1-\cos^2\alpha) + y^2(1-\sin^2\alpha) - 2xy\cos\alpha\sin\alpha + a^2(\cos^2\alpha + \sin^2\alpha) = 4ax\cos\alpha + 4ay\sin\alpha + a^2$
$x^2\sin^2\alpha + y^2\cos^2\alpha - 2xy\cos\alpha\sin\alpha + a^2$
$= 4a(x\cos\alpha + y\sin\alpha) + a^2$
$(x\sin\alpha - y\cos\alpha)^2 \cong 4a(x\cos\alpha + y\sin\alpha)$

Q.7 Show that the ordinate at any point P of the parabola is mean proportional between the length of Latus rectum and abscissa of P.
(Faisalabad Board 2019)

Sol: Let P (x, y) be a point on the parabola
$y^2 = 4ax$ then

$$y^2 = 4ax \Rightarrow \frac{y}{4a} = \frac{x}{y}$$

$$\Rightarrow \frac{4a}{y} = \frac{y}{x} \Rightarrow 4a : y = y : x$$

This show that ordinate of P is a mean proportional between the length of the latusrectum and the abscissa of P.

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 Parabola $y^2 = 4ax$, a < o opens.
(Rawalpindi Board 2013)(Sahiwal Board 2017)
(a) Upward (b) Downward
(c) Right side (d) Left side

Q.2 Focus of parabola $x^2 = -16y$ *(Sahiwal Board 2013)*
(a) (0, 4) (b) (0, −4)
(c) (4, 0) (d) (−4, 0)

Q.3 The directrix of parabola $x^2 = -16y$ is:
(Multan Board 2013, 2018 G-I)
(a) y − 1 = 0 (b) y + 1
(c) y −4 = 0 (d) y + 4 = 0

Q.4 Focus of the parabola $x^2 = 5y$:
(Multan Board 2013 G-II)
(a) $\left(0, -\frac{5}{4}\right)$ (b) $\left(0, \frac{5}{4}\right)$
(c) $\left(\frac{5}{4}, 0\right)$ (d) $\left(\frac{-5}{4}, 0\right)$

Q.5 The set of all the points in a plane which are equidistant from a fixed point and fixed line is called: *(Rawalpindi Board 2014)*
(a) Circle (b) Ellipse
(c) Parabola (d) Hyperbola

Q.6 The vertex of parabola $x^2 = 4ay$ is:
(Bahawalpur Board 2014)(Faisalabad Board 2018)
(a) (9, 0) (b) (−9, 0)
(c) (0, 0) (d) (0, −9)

Q.7 The focus of parabola $y^2 = 4ax$ is:
(Multan Board 2014 G-I)
(a) (0, a) (b) (0, −a)
(c) (a, 0) (d) (−a, 0)

Q.8 The length of the latus rectum of parabola $y^2 = 8x$ is: *(D.G.K Board 2014 G-I)*
(a) 2 (b) 4
(c) 6 (d) 8

Q.9 The vertex of parabola $y^2 = -8(x - 3)_2$ is.
(Sahiwal Board 2014)
(a) (3, 0) (b) (2, 1)
(c) (3, 1) (d) (0, 1)

Q.10 The parabola $y^2 = 4ax$ passes tharough:
(D.G.K Board 2013 G-II)
(a) (0, 0) (b) (a, 0)
(c) (a, a) (d) (0, 0)

Q.11 For parabola $(x - 1)^2 = 8(y + 2)$ vertex is:
(D.G.K Board 2012 G-II)
(a) (1, −2) (b) (1, 2)
(c) (−1, 2) (d) (−1, −2)

Q.12 The focus of parabola $y = 6x^2 - 1$ equals:
(D.G.K Board 2011)
(a) $\left(0, \frac{23}{24}\right)$ (b) $\left(0, \frac{24}{23}\right)$
(c) $\left(0, -\frac{23}{24}\right)$ (d) $\left(0, -\frac{24}{23}\right)$

Q.13 The directrix of the parabola $x^2 = 5y$ is:
(a) $x + \frac{5}{4} = 0$ (b) $y - \frac{5}{4} = 0$
(c) $y + \frac{5}{4} = 0$ (d) $x - \frac{5}{4} = 0$

Q.14 The vertex of the parabola $(x-1)^2 = 8(y+2)$ is:
(Multan Board 12 G-II, D.G.K Board 2010)
(a) $(-1,-2)$ (b) $(2,0)$
(c) $(0,0)$ (d) $(+1,-2)$

Q.15 Focus of parabola $x^2 = -16y$ equals:
(Sahiwal Board 2015)
(a) $(0,4)$ (b) $(0,-4)$
(c) $(0,16)$ (d) $(0,-16)$

Q.16 The point on the parabola which is closest to the focus is:
(Multan Board 2012, Multan Board 2015 G-II)
(a) Vertex (b) Directrix
(c) Focus (d) Origin

Q.17 The eccentricity "e" of parabola is:
(Gujranwala Board 2014)
(a) $0 < e < 1$ (b) $e = 1$
(c) $e > 1$ (d) $e = -1$

Q.18 The vertex of parabola $(x-1)^2 = 8(y+2)$ is:
(a) $(1,-2)$ (b) $(0,1)$
(c) $(-1,-2)$ (d) $(1,2)$

Q.19 Equation of axis of parabola $x^2 = 4ay$ is:
(Lahore Board 2015 G-II)
(a) $x = 0$ (b) $x = a$
(c) $y = 0$ (d) $y = a$

Q.20 The parabola $x^2 = y$ passes through point.
(Lahore Board 2015 G-I)
(a) $\left(\frac{1}{2},\frac{1}{2}\right)$ (b) $\left(\frac{1}{4},\frac{1}{2}\right)$
(c) $\left(\frac{1}{2},\frac{1}{4}\right)$ (d) $\left(\frac{1}{2},\frac{1}{2}\right)$

Q.21 Any chord passing through the focus of the parabola is called the: *(Gujranwala Board 2013)*
(a) Vertex of parabola
(b) Axis of the parabola
(c) Latus rectum of parabola
(d) Focal chord of parabola

Q.22 Focus of the parabola $x^2 = 4ay$ is:
(Gujranwala Board 2012)
(a) $(a,0)$ (b) $(-a,0)$
(c) $(0,a)$ (d) $(0,-a)$

Q.23 Equation of Directrix of parabola $y^2 = -4ax$ is:
(Lahore Board 2012 G-II)(Multan Board 2018 G-II)
(a) $x = a$ (b) $x = -a$
(c) $y = a$ (d) $y = -a$

Q.24 Directrix of $y^2 = 8x$ is: *(Bahawalpur Board 2016)*
(a) $x + 2 = 0$ (b) $x - 2 = 0$
(c) $y + 2$ (d) $y - 2 = 0$

Q.25 Focus of the parabola $y^2 = -4ax$ is:
(Faisalabad Board 2016)
(a) $(a,0)$ (b) $(-a,0)$
(c) $(0,a)$ (d) $(0,-a)$

Q.26 Focus of parabola $x^2 = -4ay$:
(Gujranwala Board 2016)
(a) $(a,0)$ (b) $(0,-a)$
(c) $(0,a)$ (d) $(-a,0)$

Q.27 The Latus rectum of a parabola $y^2 = 4ax$ is:
(Sargodha Board 2016)
(a) $x = -a$ (b) $y = -a$
(c) $x = a$ (d) $y = a$

Q.28 If the equation of the parabola is to $y^2 = 4ax$, then opening of the parabola is to the right of the:
(a) X-axis (b) $Y = x$
(c) Y-axis (d) None of these

Q.29 If the equation of the parabola is to $x^2 = 4ay$, then opening of the parabola is to -------- of the x-axis.
(a) Left (b) Upward
(c) Right (d) Downward

Q.30 The axis of the parabola $x^2 = 4ay$ is:
(a) $x = 0$ (b) $x = -a$
(c) $y = 0$ (d) $y = -a$

Q.31 The vertex of the parabola $x^2 = -4ay$ is:
(a) $(-a,0)$ (b) $(0,0)$
(c) $(0,-a)$ (d) $(0,a)$

Q.32 The graph of the parabola $y^2 = -4ax$ is symmetric about.
(a) X-axis (b) $Y = x$
(c) Y-axis (d) None of these

Q.33 The focus of the parabola $y^2 = 4ax$ is:
(a) $(-a,0)$ (b) $(0,a)$
(c) $(0,-a)$ (d) $(a,0)$

Q.34 If the focus lies on the y-axis with coordinates $F(0,-a)$ and directrix of the parabola is $y = a$, then the equation of the parabola is:
(a) $x^2 = 4ay$ (b) $y^2 = 4ax$
(c) $x^2 = -4ay$ (d) $y^2 = -4ax$

Q.35 The latus-rectum of the parabola $y^2 = -4ax$ is:
(a) $x = a$ (b) $x = -a$
(c) $y = a$ (d) $y = -a$

Q.36 The graph of the parabola $y^2 = -4ax$ lies in quadrants.
(a) I and II (b) III and IV
(c) II and III (d) I and IV

Q.37 the directrix of the parabola $y^2 = -4ax$ is:
(a) $x = a$ (b) $x = -a$
(c) $y = a$ (d) $y = -a$

Q.38 The coordinates of the end points of the latus-rectum of the parabola $y^2 = 4ax$ are (a, 2a) and ----.
(a) (−a, −2a) (b) (a, 2a)
(c) (a, −2a) (d) (−a, 2a)

Q.39 The opening of the parabola $y^2 = 4ax$ is to the ---------- of the y-axis.
(a) Left (b) Upward
(c) Right (d) Downward

Q.40 The vertex of the parabola $y^2 = 4ax$ is:
(a) (−a, 0) (b) (a, 0)
(c) (0, −a) (d) (0, 0)

Q.41 The directrix of the parabola $x^2 = 4ay$ is:
(a) x = a (b) x = −a
(c) y = a (d) y = −a

Q.42 A line joining two distinct points on a parabola is called a --------- of the parabola.
(a) Chord (b) Vertex
(c) Focus (d) Dirctrix

Q.43 The focus of the parabola $x^2 = -4ay$ is:
(a) (−a, 0) (b) (0, 0)
(c) (0, −a) (d) (0, a)

Q.44 The opening of the parabola $y^2 = -4ax$ is to the left of the
(a) x-axis (b) x = 1
(c) y-axis (d) x = 0

Q.45 The point where the axis meets the parabola is called --------- of the parabola.
(a) Directrix (b) Vertex
(c) Focus (d) Eccentricity

Q.46 The length of the latus rectum of the parabola $y^2 = 4ax$ is: *(Bahawalpur Board 2018)*
(a) a (b) 4a
(c) 2a (d) None of these

Q.47 If the focus lies on the x-axis with coordinates F(−a, 0) and directrix of the parabola is x = a, then the equation of the parabola is:
(a) $x^2 = 4ay$ (b) $y^2 = 4ax$
(c) $x^2 = -4ay$ (d) $y^2 = -4ax$

Q.48 The opening of the parabola $x^2 = 4ay$ is to ------- of the x-axis.
(a) Left (b) Upward
(c) Right (d) Downward

Q.49 The parabola $y^2 = 4ax$ lies in quadrants.
(a) I and II (b) III and IV
(c) II and III (d) I and IV

Q.50 The graph of the parabola $x^2 = -4ay$ symmetric about
(a) x-axis (b) major axis
(c) y-axis (d) minor axis

Q.51 The axis of the parabola $y^2 = -4ax$ is:
(a) x = a (b) x = 0
(c) y = a (d) y = 0

Q.52 The latus-rectum of the parabola $x^2 = -4ay$ is:
(a) x = a (b) x = −a
(c) y = a (d) y = −a

Q.53 $y^2 = 4ax$, is the standard equation of the:
(a) Ellipse (b) Parabola
(c) Hyperbola (d) None of these

Q.54 The focal chord perpendicular to the axis of the parabola is called ---------- of the parabola. *(Lahore Board 2018 G-II)*
(a) Directrix (b) Latus rectum
(c) Focus (d) Focal chord

Q.55 The graph of the parabola $x^2 = -4ay$ lies in quadrants
(a) I and II (b) III and IV
(c) II and III (d) I and IV

Q.56 The axis of the parabola $y^2 = 4ax$ is:
(a) x = 0 (b) x = a
(c) y = 0 (d) y = a

Q.57 The directrix of the parabola $x^2 = -4ay$ is:
(a) x = a (b) x = −a
(c) y = a (d) y = −a

Q.58 The vertex of the parabola $x^2 = 4ay$ is:
(a) (−a, 0) (b) (0, a)
(c) (0, −a) (d) (0, 0)

Q.59 The axis of the parabola $x^2 = -4ay$ is:
(a) x = a (b) x = 0
(c) y = a (d) y = 0

Q.60 If the focus lies on the x-axis with coordinates F(a, 0) and directrix of the parabola is x = −a, then the equation of parabola is:
(a) $x^2 = 4ay$ (b) $y^2 = 4ax$
(c) $x^2 = -4ay$ (d) $y^2 = -4ax$

Q.61 If the equation of the parabola is to $y^2 = -4ax$, then opening of the parabola is to the ---------- of the y-axy.
(a) Left (b) Upward
(c) Right (d) Downward

Q.62 If the equation of the parabola is to $x^2 = 4ay$, then opening of the parabola is to upward of the.
(a) x-axis (b) major axis
(c) y-axis (d) minor axis

Q.63 The graph of the parabola $y^2 = 4ax$ symmetric about *(Lahore Board 2018 G-I)*
(a) x-axi (b) major axis
(c) y-axis (d) minor axis

Q.64 The graph of the parabola $x^2 = 4ay$ symmetric about
(a) x-axis (b) $y = x$
(c) y-axis (d) None of these

Q.65 The directrix of the parabola $y^2 = 4ax$ is:
(a) $x = a$ (b) $x = -a$
(c) $y = a$ (d) $y = -a$

Q.66 The vertex of the parabola $y^2 = -4ax$ is:
(a) $(-a, 0)$ (b) $(a, 0)$
(c) $(0, -a)$ (d) $(0, 0)$

Q.67 The focus of the parabola $x^2 = 4ay$ is:
(a) $(-a, 0)$ (b) $(a, 0)$
(c) $(0, -a)$ (d) $(0, a)$

Q.68 The latus-rectum of the parabola $x^2 = 4ay$ is:
(a) $x = a$ (b) $x = -a$
(c) $y = a$ (d) $y = -a$

Q.69 The number e denotes the ---------- of the conic.
(a) Directrix (b) Vertex
(c) Focus (d) Eccentricity

Q.70 If the focus lies on the y-axis with coordinates F(0, a) and Directrix of the parabola is $y = -a$, then the equation of parabola is:
(a) $x^2 = 4ay$ (b) $y^2 = 4ax$
(c) $y^2 = -4ax$ (d) $x^2 = -4ay$

Q.71 Parabola having equation $x^2 = 4ay$ open. *(Sargodha Board 2017)*
(a) Towards left (b) Towards right
(c) Upwards (d) Downwards

Q.72 The directrix of the parabola $x^2 = -8y$ is. *(Lahore Board 2017 G-II)*
(a) $x + 2 = 0$ (b) $x - 2 = 0$
(c) $y + 2 = 0$ (d) $y - 2 = 0$

Q.73 The length of Latus rectum $x^2 - 4x - 3y + 13 = 0$ equals: *(Multan Board 2011 G-I)*
(a) 13 (b) 12
(c) 4 (d) 3

Q.74 The length of latus rectum of parabola $x^2 = 5y$ is. *(Gujranwala Board 2018)*
(a) 5 (b) 20
(c) $\frac{5}{4}$ (d) 10

Q.75 Focus of the Parabola $x^2 = 4ay$ is: *(Bahawalpur Board 2019)*
(a) (a , 0) (b) (−a , 0)
(c) (0 , a) (d) (0 , −a)

Q.76 The vertex of the parabola $y^2 = 16x$ is: *(Faisalabad Board 2019)*
(a) (0 , 0) (b) (1 , 0)
(c) (0 , 1) (d) (1 , 1)

Q.77 The latus rectum of the parabola $y^2 = -4ax$ is: *(Faisalabad Board 2019 G-II)*
(a) $x = a$ (b) $x = -a$
(c) $y = a$ (d) $y = -a$

Q.78 Focus of parabola $x^2 = -16y$ is *(Gujranwala Board 2019 G-I)*
(a) (0, −4) (b) (0, 4)
(c) (4, 0) (d) (−4, 0)

Q.79 Axis of parabola $x^2 = 4$ ay is *(Gujranwala Board 2019 G-II)*
(a) $x = 0$ (b) $y = 0$
(c) $y = x$ (d) $x = -y$

Q.80 Equation of latus-rectum of parabola $y^2 = 4ax$ is: *(Lahore Board 2019)*
(a) $x = -a$ (b) $y = -a$
(c) $x = a$ (d) $y = a$

Q.81 Parabola $x^2 = -8y$ opens: *(Multan Board 2019)*
(a) Rightwards (b) Leftwards
(c) Upwards (d) Downwards

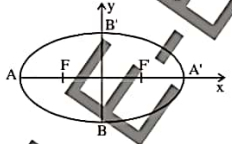
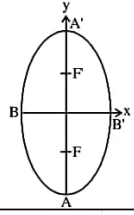
Q.82 The equation of directrix of the parabola $x^2 = 4ay$ is: *(Rawalpindi Board 2019)*
(a) $x = a$ (b) $x = -a$
(c) $y = -a$ (d) $y = a$

Q.83 The co-ordinates of vertex of Parabola $x + 8 - y^2 + 2y = 0$ will be *(Sargodha Board 2019)*
(a) (−9 , 1) (b) (9 , 1)
(c) (9 , −1) (d) (−9 , −1)

Q.84 Axis of parabola $y^2 = 4ax$ is *(Sahiwal Board 2019)*
(a) $y = 0$ (b) $x = 0$
(c) $y = a$ (d) $x = a$

Q.85 Directrix of parabola $x^2 = 20y$ is: *(Multan Board 2019)*
(a) $x = 10$ (b) $x = 5$
(c) $y = -5$ (d) $x = -5$

SUMMARY OF STANDARD ELLIPSES

Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$ $c^2 = a^2 - b^2$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b$ $c^2 = a^2 - b^2$
Focus	$(\pm c, 0)$	$(0, \pm c)$
Directrics	$x = \pm \frac{c}{e^2}$	$y = \pm \frac{c}{e^2}$
Major axis	$y = 0$	$x = 0$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Coverties	$(0, \pm b)$	$(\pm b, 0)$
Center	$(0, 0)$	$(0, 0)$
Eccentricity	$e = \frac{c}{a} < 1$	$e = \frac{c}{a} < 1$
Graph		

Note: In each ellipse

Length of major axis = 2a, Length of minor axis = 2b

Length of Latesrectum = $\frac{2b^2}{a}$, Foci lie on the major axis.

EXERCISE 6.5

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 The length of latus rectum of ellipse $\frac{(x-1)^2}{9}+\frac{(y-2)^2}{6}=1$ *(Sargodha Board 2013)*

(a) 1 (b) 2
(c) 4 (d) 12

Q.2 If a > b, then foci of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are *(Multan Board 2013 G-I)*

(a) $(\pm a, 0)$ (b) $(0, \pm a)$
(c) $(0, \pm ae)$ (d) $(\pm ae, 0)$

Q.3 The centre of the ellipse $\frac{(x+h)^2}{a^2}+\frac{(y+k)^2}{b^2})=1$. *(Multan Board 2014 G-II)*

(a) (h, k) (b) (−h, −k)
(c) (−g, k) (d) (h, −k)

Q.4 For ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a > b where $(\pm c, 0)$ are focis if: *(Sahiwal Board 2014)*

(a) $c^2 = a^2 + b^2$ (b) $c^2 = a^2 - b^2$
(c) $c^2 = -a^2 + b^2$ (d) $c^2 = -a^2 - b^2$

Q.5 The centre of ellipse $\frac{(x-1)^2}{a^2}+\frac{(y+1)^2}{b^2}=1$ is: *(D.G.K Board 2013 G-I)*

(a) (−1, −1) (b) (−5, −3)
(c) (0, 0) (d) (1, −1)

Q.6 The distance between focii of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is: *(D.G.K Board 2012 G-II)*

(a) a^2e^2 (b) ae
(c) $2ae$ (d) $2a^2e^2$

Q.7 The centre of the ellipse $x^2 + 4y^2 = 6$ is: *(D.G.K Board 2010)*

(a) (0, 0) (b) (0, 4)
(c) (1, 2) (d) (4, 0)

Q.8 The length of latus rectum of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$: *(D.G.K Board 2010) (Lahore Board 2016 G-I)*

(a) $\frac{1}{9}$ (b) $\frac{2}{9}$
(c) $\frac{9}{2}$ (d) 9

Q.9 The end points of the major axis of the ellipse are called its: *(Multan Board 2015 G-I) (A.J.K Board 2017)*

(a) Focii (b) Vertices
(c) Co-vertices (d) Directrix

Q.10 The centre of circle $x^2 + y^2 + 12x - 10y = 0$ is: *(Multan Board 2012 G-II, Multan Board 2015 G-I)*

(a) (6, 5) (b) (−6, 5)
(c) (5, 6) (d) (6, −5)

Q.11 Length of minor axis of the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$ is: *(Multan Board 2012 G-II)*

(a) 16 (b) 2
(c) 8 (d) 4

Q.12 In ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, co-ordinate (0, b) represents: *(Multan Board 2012 G-I)*

(a) Vertices (b) Co-vertices
(c) Foci (d) Origin

Q.13 The eccentricity of ellipse $\frac{x}{25}+\frac{y^2}{16}=1$ equals: *(Multan Board 2011 G-II)*

(a) $\frac{5}{4}$ (b) $\frac{4}{5}$
(c) $\frac{5}{3}$ (d) $\frac{3}{5}$

Q.14 The centre of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ equals: *(Multan Board 2011 G-I)*

(a) (0, 0) (b) (0, a)
(c) (0, b) (d) (a, b)

Q.15 Foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $c^2 = a^2 - b^2$ are: *(Lahore Board 2013 G-II)*

(a) $(\pm c, 0)$ (b) $(0, \pm c)$
(c) $(c, \pm c)$ (d) $(\pm c, c)$

Q.16 Foci of ellipse $\frac{x^2}{16}+\frac{y^2}{12}=1$ are: *(Lahore Board 2013 G-I)*

(a) $(\pm 1, 0)$ (b) $(0, \pm 1)$
(c) $(\pm 2, 0)$ (d) $(0, \pm 2)$

Q.17 Length of minor axis of $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ (a > b) is: *(Lahore Board 2012 G-II)*

(a) a (b) b
(c) 2a (d) 2b

Q.18 Length of latus rectum of $9x^2 + 4y^2 = 36$ is:
(Lahore Board 2012 G-I)

(a) $\frac{3}{8}$ (b) $\frac{8}{3}$

(c) $\frac{3}{4}$ (d) $\frac{4}{3}$

Q.19 Mid point of line segment joining foci of ellipse is called its. *(Rawalpindi Board 2016)*

(a) Centre (b) Vertex

(c) Directrix (d) Major axis

Q.20 The directrics of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b is:

(a) $x = \pm \frac{c}{e^2}$ (b) $x = \pm \frac{c}{a}$

(c) $y = \pm \frac{c}{e^2}$ (d) $y = \pm \frac{c}{a}$

Q.21 The centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b is:

(a) (a, 0) (b) (± b, 0)

(c) (0, 0) (d))0, ± a)

Q.22 The eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b is:

(a) $e = \frac{c}{a} > 1$ (b) $e = \frac{c}{a} < 1$

(c) $e = \pm \frac{e}{a}$ (d) None of these

Q.23 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = I$ is an equation of the ---------- in standard form. *(Bahawalpur Board 2018)*

(a) Ellipse (b) Parabola

(c) Hyperbola (d) None of these

Q.24 The foci of the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, a > b are:

(a) (0, ± c) (b) (± c, 0)

(c) (± a, 0) (d) (0, ± a)

Q.25 Eccentricity of the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, a > b is:

(a) e > 1 (b) e < 1

(c) e = 0 (d) None of these

Q.26 The vertices of the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, a > b is:

(a) (0, ± a) (b) (± c, 0)

(c) (± a, 0) (d) (0, ± c)

Q.27 The length of major axis of the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, a > b is:

(a) 4a (b) 2a

(c) 4b (d) 2b

Q.28 The major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b is:

(a) x = 0 (b) x = −a

(c) y = 0 (d) y = −a

Q.29 The length of minor axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b is: *(Multan Bord 2017 G-I)*

(a) a (b) 2a

(c) b (d) 2b

Q.30 The covertices of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b is:

(a) (± b, 0) (b) (0, ± b)

(c) (± a, 0) (d) (0, ± a)

Q.31 The centre of the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, a > b is:

(a) (0, 0) (b) (± c, 0)

(c) (a, 0) (d) (0, a)

Q.32 Foci of ellipse lie on the

(a) x-axis (b) major axis

(c) y-axis (d) minor axis

Q.33 The major axis of the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, a > b is:

(a) x = a (b) x = 0

(c) y = a (d) y = 0

Q.34 The vertices of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b is:
(Faisalabad Board 2018)

(a) (0, ± b) (b) (± b, 0)

(c) (± a, 0) (d) (0, ± a)

Q.35 The directrics of the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, a > b is:

(a) $x = \pm \frac{c}{e^2}$ (b) $x = \pm \frac{c}{a}$

(c) $y = \pm \frac{c}{e^2}$ (d) $y = \pm \frac{c}{a}$

Q.36 The covertices of the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, a > b is:

(a) (0, ± b) (b) (± b, 0)

(c) (± a, 0) (d) (0, ± a)

Q.37 A line that touches the curve without cutting through it is called.

(a) Straight line (b) Tangent line
(c) Normal line (d) None of these

Q.38 The length of Latus rectum of the ellipse. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is. *(Faisalabad Board 2017, 2018)*

(Lahore Board 2017 G-I)

(a) $\frac{2b^2}{a}$ (b) $\frac{a^2}{2b^2}$

(c) $\frac{a}{2b^2}$ (d) 2ab

Q.39 Vertices of ellipse with equation $x^2 + 4y^2 = 16$ are. *(Sargodha Board 2017)*

(a) $(\pm 4, 0)$ (b) $(0, \pm 4)$
(c) $(\pm 2, 0)$ (d) $(0, \pm 2)$

Q.40 Eccentricity of an ellipse is.

(D.G.K Board 2017 G-I)(Rawalpindi Board 2017 G-II)

(a) $e = 1$ (b) $e > 1$
(c) $0 < e < 1$ (d) $e = 0$

Q.41 Vertices of the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ are.

(Sargodha Board 2018)

(a) $(0, \pm 5)$ (b) $(\pm 5, 0)$
(c) $(\pm 4, 0)$ (d) $(0, \pm 4)$

Q.42 Length of major and minor axes of ellipse $x^2 + 16y^2 = 16$ is. *(Sahiwal Board 2018)*

(a) 4, 1 (b) 10, 5
(c) 8, 2 (d) 16, 2

Q.43 The length of latus rectum of ellipse $\frac{x^2}{36} + \frac{y^2}{25} = 1$. *(Sahiwal Board 2018)*

(a) $\frac{25}{6}$ (b) $\frac{25}{3}$
(c) $\frac{25}{36}$ (d) $\frac{3}{25}$

Q.44 For ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ then eccentricity e =

(Lahore Board 2018 G-II)

(a) $\frac{\sqrt{a^2 - b^2}}{a}$ (b) $\frac{\sqrt{a^2 + b^2}}{a}$
(c) $\frac{\sqrt{b^2 - a^2}}{a}$ (d) $\frac{\sqrt{b^2 - a^2}}{b}$

Q.45 If a = b, then equation $\frac{x^2}{a^2} + \frac{b^2}{b^2} = 1$ represents.

(Gujranwala Board 2018)

(a) Ellipse (b) Parabola
(c) Hyperbola (d) Circle

Q.46 Focus of $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is.

(Multan Board 2018 G-II)

(a) $(\pm 4, 0)$ (b) $(\pm 5, 0)$
(c) $(0, \pm 3)$ (d) $(\pm 3, 0)$

Q.47 The end-points of minor axis of an ellipse are called: *(Faisalabad Board 2019)*

(a) Foci (b) Vertices
(c) Covertices (d) Centre

Q.48 Eccentricity of an ellipse is

(Gujranwala Board 2019 G-II)

(a) $e = 1$ (b) $e > 1$
(c) $0 < e < 1$ (d) $e = 0$

Q.49 The centre of Ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 16$ is:

(Rawalpindi Board 2019)

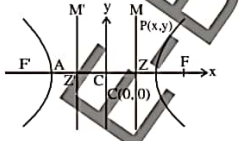
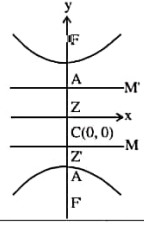
(a) (4 , 1) (b) (1 , 4)
(c) (−1 , 4) (d) (0 , 0)

Q.50 For ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; a^2e^2 will be equal to:

(Sargodha Board 2019)

(a) $b^2 - a^2$ (b) $a^2(1 - b^2)$
(c) $a^2 - b^2$ (d) $a^2 + b^2$

SUMMARY OF STANDARD HYPERBOLAS

Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $c^2 = a^2 + b^2$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ $c^2 = a^2 + b^2$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Directrices	$x = \pm \frac{c}{e^2}$	$y = \pm \frac{c}{e^2}$
Transverse axis	$y = 0$	$x = 0$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Eccentricity	$e = \frac{c}{a} > 1$	$e = \frac{c}{a} > 1$
Center	(0, 0)	(0, 0)
Asymptotes	$y = \pm \frac{b}{a} x$	$y = \pm \frac{a}{b} x$
Graph		

EXERCISE 6.6

SHORT ANSWERS TO THE QUESTIONS

Q.1 Define hyperbola. *(Lahore Board 2013) (Sahiwal Board 2013)*

Ans. Hyperbola:

Let F be a fixed point and L be a fixed line not containing F. If P(x, y) be any point in the plane such that |PM| is the ⊥ distance of P from L, then set of all points P(x,y).

Such that $\frac{|PF|}{|PM|} = e > 1$ is called a hyperbola. F is called focus and 'e' is called eccentricity of hyperbola.

Q.2 Define asymptote.

Ans. A line, which touches a given curve and the point of contact, is at infinity.

Q.3 Find the foci and eccentricity of the hyperbola

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

Ans. Given equation can be written as

$$\frac{(y-0)^2}{4^2} - \frac{(x-0)^2}{3^2} = 1 \quad \ldots\ldots\ldots (1)$$

Here $a = 4, b = 3$

Since $c^2 = a^2 + b^2 \Rightarrow c^2 = 16 + 9 = 25 \Rightarrow c = 5$

Eccentricity $= e = \frac{c}{a} = \frac{5}{4}$

The transverse axis of (1) lies along the y-axis

Coordinate of foci are $(0, \pm 5)$

Q.4 Find the center, vertices, foci and eccentricity and equations directrices of the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$ *(Gujranwala Board 2011, 2017) (Multan Board 2013 G-I)(Sargodha Board 2019) (Faisalabad Board 2019 G-I)*

Ans. Given equation can be written as

$$\frac{(x-0)^2}{2^2} - \frac{(y-0)^2}{3^2} = 1 \quad \ldots\ldots\ldots (1)$$

Center: C(0, 0)

Here $a = 2, b = 3$

$c^2 = 4 + 9 = 13 \Rightarrow c = \sqrt{13}$

Eccentricity $= e = \frac{c}{a} = \frac{\sqrt{13}}{2}$

Vertices: $(\pm 2, 0)$

Equations of directrices are $x = \pm \frac{a}{2}$

The transverse axis of (1) lies along the x-axis

Coordinates of foci are $(\pm\sqrt{13}, 0)$

Q.8 Find the foci, vertices and eccentricity of the hyperbola

$$\frac{y^2}{16} - \frac{x^2}{49} = 1 \ldots\ldots (1)$$ *(Sargodha Board 2016)*

Ans. The transverse axis of (1) lies along the y-axis

Coordinates of the vertices are $(0, \pm 4)$

Here $a = 4, b = 7$ so that from $c^2 = a^2 + b^2$, we get

$c^2 = 16 + 49$ or $c = \sqrt{65}$

Foci are: $(0, \pm\sqrt{65})$

Eccentricity: $= \frac{c}{a} = \frac{\sqrt{65}}{4}$

Vertices: $(0, \pm 4)$

LONG QUESTIONS

Q.1 Find the center, foci, eccentricity and equations of directrices of hyperbola.

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$ *(Bahawalpur Board 2019) (Gujranwala, Sargodha Board 2018)*

Sol: See Short Question 7

Q.2 Find the center, foci, eccentricity vertices and equations of directrices of $\frac{x^2}{16} - \frac{y^2}{9} = 1$ *(Sahiwal Board 2018)*

Sol: See Short Question 5

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 Foci of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are: *(D.G.K Board 2014 G-I)*

(a) $(\pm a, 0)$ (b) $(0, \pm a)$

(c) $(0, \pm ae)$ (d) $(\pm ae, 0)$

Q.2 Asymptotes are very useful in graphing. *(D.G.K Board 2015 G-I, D.G.K Board 2014 G-II)*

(a) Circle (b) Parabola

(c) Ellipse (d) Hyperbola

Q.3 Vertices of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are: *(D.G.K Board 2013 G-I)(Rawalpindi Board 2017 G-I) (Multan Board 2011 G-II)*

(a) $(\pm a, 0)$ (b) $(0, \pm a)$

(c) $(0, \pm ae)$ (d) $(\pm ae, 0)$

Q.4 For hyperbola. *(Lahore Board 2012 G-I)*

(a) $c^2 = a^2 - b^2$ (b) $c^2 = a^2 + b^2$

(c) $b^2 = a^2 + c^2$ (d) $a^2 + b^2 + c^2$

Q.5 In equation of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{a^2} = 1$ the value of c^2 = ---------

(a) $a^2 + b^2$ (b) $b^2 - a^2$
(c) $a^2 - b^2$ (d) $b^2 - a^2$

Q.6 The lines ---------- are called asymptotes of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

(a) $x = \pm \frac{b}{a} y$ (b) $y = \pm \frac{a}{b} x$

(c) $x = \pm \frac{b}{a} y$ (d) $y = \pm \frac{b}{a} x$

Q.7 The transverse axis of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is:

(a) $x = 0$ (b) $x = a$
(c) $y = 0$ (d) $y = -a$

Q.8 The transverse axis of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is:

(a) $x = \pm a$ (b) $x = 0$
(c) $y = \pm b$ (d) $y = 0$

Q.9 The eccentricity "e" of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is:

(a) $e = \frac{c}{a} > 1$ (b) $e = \frac{c}{a} < 1$

(c) $e = \pm \frac{c}{a}$ (d) None of these

Q.10 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the standard equation of the

(a) Ellipse (b) Parabola
(c) Hyperbola (d) None of these

Q.11 The conic is a hyperbola, if

(a) $e = 1$ (b) $e > 1$
(c) $0 < e < 1$ (d) None of these

Q.12 For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, c^2 =

(a) $a^2 + b^2$ (b) $b^2 - a^2$
(c) $a^2 - b^2$ (d) $b^2 - a^2$

Q.13 The directrics of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

(a) $y = \pm \frac{c}{a}$ (b) $x = \pm \frac{c}{a}$

(c) $y = \pm \frac{c}{e^2}$ (d) $x = \pm \frac{c}{e^2}$

Q.14 The centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

(a) (0, 0) (b) (± c, 0)
(c) (± a, 0) (d) (0, ± a)

Q.15 The vertices of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is

(a) (0, ± c) (b) (± c, 0)
(c) (± a, 0) (d) (0, ± a)

Q.16 The directrices of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ are

(a) $x + \pm \frac{c}{e^2}$ (b) $x = + \frac{c}{a}$

(c) $y = \pm \frac{c}{e^2}$ (d) $y = \pm \frac{c}{a}$

Q.17 The foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

(a) (0, ± c) (b) (± c, 0)
(c) (± c, ± c) (d) (0, 0)

Q.18 A conic section is hyperbola if eccentricity 'e' is
(Lahore Board 2013 G-II)(Gujranwala Board 2018)

(a) $e = 0$ (b) $e = 1$
(c) $0 < e < 1$ (d) $e > 1$

Q.19 The eccentricity of $\frac{y^2}{4} - x^2 = 1$ equals.
(Rawalpindi Board 2018)

(a) $\frac{2}{\sqrt{5}}$ (b) $\frac{-2}{\sqrt{5}}$

(c) $\frac{\sqrt{5}}{2}$ (d) $\frac{-\sqrt{5}}{2}$

Q.20 The two separate parts of hyperbola are called:
(Faisalabad Board 2019 G-II)

(a) Foci (b) Vertices
(c) Directrices (d) Branches

EXERCISE 6.7

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find the equation of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (a cos θ, b sin θ)

Ans. Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ..... (1)

Differentiating w.r.t. x

$$\frac{1}{a^2} \cdot 2x + \frac{1}{b^2} \cdot 2y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{2y}{b^2} \cdot \frac{dy}{dx} = -\frac{2x}{a^2} \Rightarrow \frac{dy}{dx} = -\frac{xb^2}{ya^2}$$

$$\left[\frac{dy}{dx}\right]_{(a\cos\theta,\, b\sin\theta)} = -\frac{a\cos\theta \cdot b^2}{b\sin\theta \cdot a^2} = -\frac{b\cos\theta}{a\sin\theta}$$

is the slope of tangent line

Slope of normal line at P = $\frac{a\sin\theta}{b\cos\theta}$

Now equation of tangent at P is

$$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$\frac{y}{b}\sin\theta - \sin^2\theta = -\frac{x}{a}\cos\theta + \cos^2\theta$$

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = \cos^2\theta + \sin^2\theta$$

or $$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Q.2 Find the equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (a sec θ, b tan θ)

Ans. Given $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ..... (1)

Differentiating w.r.t. x, we have

$$\frac{1}{a^2}2x - \frac{1}{b^2}2y\frac{dy}{dx} = 0 \Rightarrow \frac{2y}{b^2}\frac{dy}{dx} = \frac{2x}{a^2} \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

$$\left[\frac{dy}{dx}\right]_{(a\sec\theta,\, b\tan\theta)} = \frac{b^2(a\sec\theta)}{a^2(b\tan\theta)} = \frac{b\sec\theta}{a\tan\theta}$$

is the slope of tangent line

Slope of tangent line at P $= -\frac{a\tan q}{b\sec q}$

Now equation of tangent at P is

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$\frac{\tan q}{b}(y - b\tan\theta) = \frac{\sec q}{a}(x - a\sec\theta)$$

$$\frac{y}{b}\tan\theta - \tan^2\theta = \frac{x}{a}\sec\theta - \sec^2\theta$$

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = \sec^2\theta - \tan^2\theta$$

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \quad \{\because \sec^2\theta - \tan^2\theta = 1\}$$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 Equation of the tangent to the circle $x^2 + y^2 = 4$ at $(\sqrt{3}, 1)$ is: *(Multan Board 2012 G-I)*

(a) $x + \sqrt{3}y = 4$ (b) $x - \sqrt{3}y = 4$

(c) $\sqrt{3}x - y = 4$ (d) $\sqrt{3}x + y = 4$

Q.2 Equation of tangent to circle $x^2 + y^2 = a^2$ at $P(x_1, y_1)$ is: *(Bahawalpur Board 2016)*

(a) $xx_1 + yy_1 = a^2$ (b) $xx_1 - yy_1 = a^2$

(c) $xy_1 + yx_1 = a^2$ (d) $xy_1 - yx_1 = a^2$

Q.3 The line $y = mx + c$ will be tangent to the parabola $y^2 = 4ax$ if: *(Lahore Board 2016 G-I)*

(a) $c = -am^2$ (b) $c = \frac{a}{m}$

(c) $c = a(1 + m^2)$ (d) $c = \frac{m}{a}$

Q.4 The parametric equations of the circle $x^2 + y^2 = r^2$ are:

(a) $x = r\sin\theta$, $y = r\sin\theta$

(b) $x = r\cos\theta$, $y = r\sin\theta$

(c) $x = r\cos\theta$, $y = r\cos\theta$

(d) None of these

Q.5 Equation of normal line to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1) is: *(Gujranwala Board 2012)*

(a) $xy_1 = x_1y$ (b) $xy_1 = -x_1y$

(c) $xx_1 + yy_1 = 0$ (d) $xx_1 + yy_1 = a^2$

Q.6 Equation of tangent to the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ at (2, 2) is: *(D.G.K Board 2012 G-I)*

(a) $\frac{x}{2} + y = 1$ (b) $x + \frac{y}{2} = 1$

(c) $2x + y = 1$ (d) None

Q.7 The straight line $y = mx + c$ is tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if. *(Lahore Board 2018 G-I)*

(a) $c^2 = a^2m^2 - b^2$ (b) $c^2 = b^2m^2 + a^2$

(c) $c^2 = b^2m^2 - a^2$ (d) $c^2 = a^2m^2 + b^2$

Q.8 Slope of tangent to parabola $y^2 = 4ax$ at (a, 2a) is: *(Bahawalpur Board 2019)*

(a) 3 (b) 2

(c) −1 (d) 1

Q.9 Slope of tangent to the curve $x^2 - y^2 - 12 = 0$ at point (4, 2) will be equal to *(Sargodha Board 2019)*

(a) 4 (b) $\frac{1}{4}$

(c) 2 (d) $\frac{1}{2}$

Q.10 The equation of tangent drawn from point (2 , 1) to the circle $x^2 + y^2 + 2y = 0$ will be

(a) $2y + x = 0$ (b) $2y - x = 0$

(c) $2x - y = 0$ (d) $2x + y - 5 = 0$

EXERCISE 6.8

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 ----------, $y = x \sin\theta + Y \cos\theta$ are equations of transformation for a rotation of axes through an angle θ, $(0 < \theta < 90°)$.

(a) $x = X \cos\theta + Y \sin\theta$

(b) $x = X \sin\theta - Y \cos\theta$

(c) $x = X \cos\theta - Y \sin\theta$

(d) None of these

Q.2 The general equation of second degree $Ax^2 + By^2 + Gx + fy + c = 0$ is a\an ---------- if $A \neq B$ and both are of opposite signs.

(a) Ellipse (b) Parabola

(c) Hyperbola (d) Circle

Q.3 The general equation of second degree $Ax^2 + By^2 + Gx + Fy + c = 0$ is a\an ---------- if $A \neq B$ and both are of the same signs.

(a) Parabola (b) Ellipse

(c) Hyperbola (d) Circle

Q.4 The general equation of second degree $Ax^2 + By^2 + Gx + Fy + c = 0$ is a\an ------ if $A = B \neq 0$.

(a) Circle (b) Parabola

(c) Hyperbola (d) Ellipse

Q.5 The general equation of second degree. $ax^2 + by^2 + 2hxy + 2fy + c = 0$ is a parabola if

(a) $h^2 - ab = 0$ (b) $h^2 - ab \neq 0$

(c) $h^2 - ab < 0$ (d) $h^2 - ab > 0$

Q.6 The general equation of second degree. $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ is an ellipse or a circle, if

(a) $h^2 - ab = 0$ (b) $h^2 - ab \neq 0$

(c) $h^2 - ab < 0$ (d) $h^2 - ab > 0$

Q.7 The general equation of second degree $Ax^2 + By^2 + Gx + Fy + c = 0$ is a\an ---------- if either $A = 0$ or $B = 0$.

(a) Ellipse (b) Parabola

(c) Hyperbola (d) Circle

Q.8 The general equation of second degree. $ax^2 + by^2 + 2hxy + 2fy + c = 0$ is a hyperbola if

(a) $h^2 - ab = 0$ (b) $h^2 - ab < 0$

(c) $h^2 - ab \neq 0$ (d) $h^2 - ab > 0$

Q.9 The axes are rotated about the origin through an angle θ of $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ is given by ----------, where $0 < \theta < 90°$.

(a) $\tan 2\theta = \frac{2f}{a-b}$ (b) $\tan 2\theta = \frac{2g}{a-b}$

(c) $\tan 2\theta = \frac{2h}{a-b}$ (d) None of these

EXERCISE 7.1

SHORT ANSWERS TO THE QUESTIONS

Q.1 Define Scalar quantities.

Ans. Physical quantities, which has magnitude but no direction. The examples are time, population, mass, volume, density etc.

Q.2 Define Vector quantities.

Ans. Physical quantities, which has magnitude as well as direction. The examples are displacement, velocity, acceleration, force, weight etc.

Q.3 Define a Unit vector. ***(Lahore Board 2018 G-I)***

Ans. A vector whose magnitude is unity is called a unit vector. The unit vector $\hat{a}$ in the direction of $\vec{a}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

Q.4 Define a Null vector.

Ans. A vector of zero magnitude is called a null vector. It has no specific direction.

Q.5 Define Position Vector.

(Rawalpindi Board 2019)

Ans. A vector, whose initial point is the origin O and whose terminal point is P, is called the position vector of P and is written as $\overrightarrow{OP}$.

Q.6 Find the magnitude of the vector $\vec{u} = [3, -4]$

(Sahiwal Board 2013)

Ans. $\vec{u} = [3, -4]$

$\vec{u} = 3i - 3j$

$|\vec{u}| = \sqrt{(3)^2 + (-4)^2}$

$|\vec{u}| = \sqrt{9 + 16}$

$|\vec{u}| = \sqrt{25} = 5$

Q.7 Find the unit vector in the same direction as the vector $\underline{v} = [3, -4]$

Ans. Given: $\underline{v} = [3, -4] \Rightarrow |\underline{v}| = \sqrt{(3)^2 + (-4)^2}$

$= \sqrt{9+16} = \sqrt{25} = 5$

Now unit vector of $\underline{v} = \hat{v} = \frac{\underline{v}}{|\underline{v}|}$

$= \frac{1}{5}[3, -4] = \left[\frac{3}{5}, \frac{-4}{5}\right] = \frac{3}{5}\underline{i} - \frac{4}{5}\underline{j}$

Q.8 Find a unit vector in the direction of the vector $\underline{v} = 2\underline{i} + 6\underline{j}$ ***(Multan Board 2017 G-I, 2018 G-II)***

(D.G Khan Board 2017 G-I)(Sargodha Board 2019)

Ans. $\underline{v} = 2\underline{i} + 6\underline{j}$

$|\underline{v}| = \sqrt{(2)^2 + (6)^2} = \sqrt{4 + 36} = \sqrt{40}$

$\therefore$ A unit vector in the direction of $\underline{v}$

$= \frac{\underline{v}}{|\underline{v}|} = \frac{1}{\sqrt{40}}(2\underline{i} + 6\underline{j}) = \frac{1}{2\sqrt{10}}(2\underline{i} + 6\underline{j})$

$= \frac{1}{\sqrt{10}}\underline{i} + \frac{3}{\sqrt{10}}\underline{j}$

Q.9 Find a unit vector in the direction of the vector $\underline{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$ ***(Lahore Board 2018 G-I)***

(D.G Khan Board 2017 G-II)(Rawalpindi Board 2019)

Ans. $|\underline{v}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$

Unit vector $= \frac{\vec{U}}{|\vec{U}|} = \frac{\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{i}}{1} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$

Q.10 Find a unit vector in the direction of the vector $\underline{v} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$ ***(Lahore Board 2010 G-II)***

Ans. $\underline{v} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$

$|\underline{v}| = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{\frac{4}{4}} = 1$

$\therefore$ Unit vector in the direction of

$\underline{v} = \hat{v} = \frac{\underline{v}}{|\underline{v}|} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$

Q.11 Find a unit vector in the direction of the vector $\underline{v} = 2\underline{i} - \underline{j}$ ***(Sahiwal Board 2014 G-II)***

Ans. $\underline{v} = 2\underline{i} - \underline{j}$

$|\underline{v}| = \sqrt{(2)^2 + (-1)^2} = \sqrt{4+1} = \sqrt{5}$

$\therefore$ Unit vector in the direction of $\underline{v}$

$= \hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{1}{\sqrt{5}}(2\underline{i} - \underline{j}) = \frac{2}{\sqrt{5}}\underline{i} - \frac{1}{\sqrt{5}}\underline{j}$

Q.12 Find the sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$, given the four points
$A(1,-1)$, $B(2,0)$, $C(-1,3)$ and $D(-2,2)$.
(Gujranwala Board 2012)(Lahore Board 2014 G-II) (Faisalabad Board 2019 G-II)

Ans. Given $A(1,-1)$, $B(2,0)$, $C(-1,3)$, $D(-2,2)$

$\overrightarrow{OA} = \underline{i} - \underline{j}$, $\overrightarrow{OB} = 2\underline{i}$,

$\overrightarrow{OC} = -\underline{i} + 3\underline{j}$, $\overrightarrow{OD} = -2\underline{i} + 2\underline{j}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\underline{i} - (\underline{i} - \underline{j}) = 2\underline{i} - \underline{i} + \underline{j} = \underline{i} + \underline{j}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = (-2\underline{i} + 2\underline{j}) - (-\underline{i} + 3\underline{j})$

$= -2\underline{i} + 2\underline{j} + \underline{i} - 3\underline{j} = -\underline{i} - \underline{j}$

Now $\overrightarrow{AB} + \overrightarrow{CD} = \underline{i} + \underline{j} - \underline{i} - \underline{j} = 0$

Q.13 If O is the origin and $\overrightarrow{OP} = \overrightarrow{AB}$, find the point P when A and B are (–3, 7) and (1, 0) respectively? *(Gujranwala Board 2019 G-I) (D.G Khan Board 2014 G-I)(Lahore Board 2016 G-I)*

Ans. Given: $A(-3,7)$ and $B(1,0)$. Let $P(x,y)$.

Since $\overrightarrow{OP} = \overrightarrow{AB}$

$\overrightarrow{OP} = \overrightarrow{OB} - \overrightarrow{OA}$

$x\underline{i} + y\underline{j} = \underline{i} - (-3\underline{i} + 7\underline{j})$

$x\underline{i} + y\underline{j} = \underline{i} + 3\underline{i} - 7\underline{j}$

$x\underline{i} + y\underline{j} = 4\underline{i} - 7\underline{j}$

$x = 4$ and $y = -7$

Thus the coordinates of the point P are $(4,-7)$

Q.14 If $\overrightarrow{AB} = \overrightarrow{CD}$, find the coordinates of the point A when points B, C, D are (1, 2), (–2, 5), (4, 11) respectively.
(Multan Board 2014 G-I)(Bahawalpur Board 2019)

Ans. Given: $B(1,2)$, $C(-2,5)$, $D(4,11)$

Let coordinates of A are (x,y)

Since $\overrightarrow{AB} = \overrightarrow{CD}$

$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OD} - \overrightarrow{OC}$

$(\underline{i} + 2\underline{j}) - (x\underline{i} + y\underline{j}) = (4\underline{i} + 11\underline{j}) - (-2\underline{i} + 5\underline{j})$

$\underline{i} + 2\underline{j} - x\underline{i} - y\underline{j} = 4\underline{i} + 11\underline{j} + 2\underline{i} - 5\underline{j}$

$(1-x)\underline{i} + (2-y)\underline{j} = 6\underline{i} + 6\underline{j}$

$1 - x = 6$ and $2 - y = 6$

$-x = 6 - 1$ and $-y = 6 - 2$

$-x = 5$ and $-y = 4$

$x = -5$ and $y = -4$

Thus coordinates of A are $(-5,-4)$.

Q.15 Find a vector from the point A to the origin where $\overrightarrow{AB} = 4\underline{i} - 2\underline{j}$ and B is the point (–2, 5).
(Rawalpindi Board 2013)(Bahawalpur Board 2016) (Lahore Board 2014, 2017 G-I) (Faisalabad Board 2016, 2019 G-I)

Ans. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{AB}$

$\overrightarrow{OA} = -2\underline{i} + 5\underline{j} - 4\underline{i} + 2\underline{j}$

$\overrightarrow{OA} = -6\underline{i} + 7\underline{j}$

So, vector from A to origin is.

$\overrightarrow{OA} = -6\underline{i} + 7\underline{j}$

$\overrightarrow{OA} = -\overrightarrow{OA} = 6\underline{i} - 7\underline{j}$

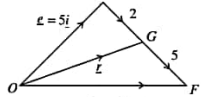
Q.16 Find position vector of a point which divide the join of point E with position vector $5\underline{i}$ and point F with position vector $4\underline{i} + \underline{j}$ in the ratio 2 : 5.
(Lahore Board 2015 G-II, 2013, 2014 G-I)

Ans. Let $\underline{e} = \overrightarrow{OE} = 5\underline{i}$, $\underline{f} = \overrightarrow{OF} = 4\underline{i} + \underline{j}$ and $p : q = 2 : 5$

By using ratio formula

$\underline{r} = \dfrac{q\underline{e} + p\underline{f}}{q + p}$

$\underline{r} = \dfrac{5(5\underline{i}) + 2(4\underline{i} + \underline{j})}{5 + 2}$



$= \dfrac{25\underline{i} + 8\underline{i} + 2\underline{j}}{7}$

$\underline{r} = \dfrac{33\underline{i} + 2\underline{j}}{7} = \dfrac{33}{7}\underline{i} + \dfrac{2}{7}\underline{j}$

Q.17 Find position vector of a point which divide the join of P and Q with position vectors $2\underline{i} - 3\underline{j}$ and $3\underline{i} + 2\underline{j}$ in ratio 4 : 3.
(Gujranwala Board 2016)

Ans. Here, $OP = 2\underline{i} - 3\underline{j}$, $OQ = 3\underline{i} + 2\underline{j}$

and $p : q = 4 : 3$

$\underline{r} = \dfrac{3(2\underline{i} - 3\underline{j}) + 4(3\underline{i} + 2\underline{j})}{3 + 4} = \dfrac{6\underline{i} - 9\underline{j} + 12\underline{i} + 8\underline{j}}{7}$

$= \dfrac{18\underline{i} - \underline{j}}{7} = \dfrac{18}{7}\underline{i} - \dfrac{1}{7}\underline{j}$

LONG QUESTIONS

Q.1 Find the vector from the point A to the origin where $\overrightarrow{AB} = 4\hat{i} - 2\hat{j}$ and B is the point (–2,5). *(Multan Board 2013 G-II, 2016 G-I)*

Sol: See Short Question 25

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 The unit vector of a vector $\vec{V}$ is: *(Faisalabad Board 2013)*

(a) $\frac{\vec{V}}{|\vec{V}|}$ (b) $|\vec{V}|$

(c) $\frac{|V|}{V}$ (d) $\frac{V}{|V|^2}$

Q.2 The magnitude of vector is also called its: *(Sargodha Board 2013)*

(a) Parameter (b) Variables

(c) Point (d) Norm

Q.3 Unit vector in the direction of $\underline{v} = 2\underline{i} - \underline{j}$ is: *(Sahiwal Board 2013)*

(a) $\frac{2\underline{i} - \underline{j}}{2}$ (b) $\frac{2\underline{i} - \underline{j}}{\sqrt{2}}$

(c) $\frac{2\underline{i} - \underline{j}}{\sqrt{3}}$ (d) $\frac{2\underline{i} - \underline{j}}{\sqrt{5}}$

Q.4 Magnitude of the vector $\underline{V} = [3, -4]$ *(Multan Board 2013 G-II)*

(a) 3 (b) 4

(c) 5 (d) 6

Q.5 If P = (a, b), Q = (a_2, b_2) then $\overrightarrow{PQ}$ is:

(a) $(a_1 + a_2)\underline{i} + (b_1 + b_2)\underline{j}$ (b) $(a_1 - a_2)\underline{i} + (b_1 - b_2)\underline{j}$

(c) $(a_2 - a_1)\underline{i} + (b_2 - b_1)\underline{j}$ (d) $(a_1 + b_1)\underline{i} + (a_2 + b_2)\underline{j}$

Q.6 If p ≡ (2, 3) and Q(6, −2), then $\overrightarrow{PQ}$ = *(D.G.K Board 2013, G-II, Multan Board 2014 G-II)*

(a) $4\underline{i} + 5\underline{j}$ (b) $-4\underline{i} + 5\underline{j}$

(c) $4\underline{i} - 5\underline{j}$ (d) $5\underline{i} - 4\underline{j}$

Q.7 The direction cosine of x-axis are: *(D.G.K Board 2015 G-II)*

(a) (1, 0, 0) (b) (1, 0, 1)

(c) (1, 1, 0) (d) (0, 0, 1)

Q.8 A vector whose magnitude is unity called: *(D.G.K Board 2013 G-II, Lahore Board 2015 G-I)*

(a) Unit vector (b) Null vector

(c) Zero vector (d) Constant vector

Q.9 If $\vec{v} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$, then $|\underline{v}|$ = *(Rawalpindi Board 2017 G-II)*

(a) 1 (b) 0

(c) $\frac{1}{2}$ (d) 4

Q.10 If $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, then $\overrightarrow{AB}$ is: *(D.G.K Board 2012 G-II)*

(a) $\vec{a} - \vec{b}$ (b) $\vec{b} - \vec{a}$

(c) $\vec{a} + \vec{b}$ (d) $\vec{a} \cdot \vec{b}$

Q.11 If $\underline{U} = k\underline{V}$, then vector $\underline{U}$ and $\underline{V}$ are in the same direction when: *(D.G.K Board 2010)*

(a) $K < 0$ (b) $\underline{K} = 0$

(c) $K > 0$ (d) $K \neq 0$

Q.12 If $\underline{i} - 3\underline{j} + \underline{k}$ and $\lambda\underline{i} + 6\underline{j} - 2\underline{k}$ are parallel, then λ equals: *(Sahiwal Board 2015)*

(a) −2 (b) 2

(c) −3 (d) 3

Q.13 Unit vector of $2\underline{i} + \underline{j}$ is: *(Multan Board 2015 G-I)*

(a) $2\underline{i} - \underline{j}$ (b) $\frac{2\underline{i} + \underline{j}}{5}$

(c) $\frac{2\underline{i} + \underline{j}}{3}$ (d) $\frac{2\underline{i} + \underline{j}}{\sqrt{5}}$

Q.14 The norm of $\underline{U} = [x_1y]$ in R^2 is: *(Lahore Board 2013 G-II)*

(a) $x^2 + y^2$ (b) $\sqrt{x^2 + y^2}$

(c) $x^2 - y^2$ (d) $\sqrt{x^2 - y^2}$

Q.15 Magnitude of the vector $\vec{V} = -\underline{i} + \underline{j}$ is: *(Lahore Board 2012 G-II)*

(a) $\sqrt{1}$ (b) $\sqrt{2}$

(c) $\sqrt{3}$ (d) $\sqrt{4}$

Q.16 A scalar quantity, is one that possesses only

(a) Magnitude (b) Direction

(c) Both a and b (d) None of these

Q.17 Velocity, acceleration, force, weight are examples of

(a) Vectors (b) Scalars

(c) Unit vectors (d) Both a and b

Q.18 Time, density, temperature, length, volume, speed and work are examples of

(a) Vectors (b) Scalars

(c) Unit vectors (d) None of these

Q.19 The law of parallelogram of addition was used by Aristotle to describe the combined action of

(a) One force (b) Two forces

(c) Three forces (d) None of these

Q.20 For any two vectors $\underline{u} = [x, y]$ and $\underline{v} = [x', y']$, $\underline{u} + \underline{v}$ =

(a) $[x + y, x' + y']$ (b) $[x + y, x' + y']$

(c) $[x + x', y + y']$ (d) None of these

Q.21 If $\overrightarrow{AB} = \underline{v}$, then $|-\overrightarrow{AB}|$ --------

(a) $-\underline{v}$ (b) $\underline{v}$

(c) $|\underline{v}|$ (d) $-|\underline{v}|$

Q.22 Two vectors are said to be equal, if they have ------ magnitude and ---------- direction.
(a) Same, same (b) Opposite, same
(c) Same, opposite (d) Opposite, opposite

Q.23 If $\underline{u}$ and $\underline{v}$ are two vectors, then difference of these two vectors are $\underline{u} + (-\underline{v})$ = ----------
(a) $\underline{v} - \underline{u}$ (b) $\underline{u} - \underline{v}$
(c) $\underline{u} + \underline{v}$ (d) 0

Q.24 Let A and B be two points whose position vectors are $\underline{a}$ and $\underline{b}$ respectively. If a point P divides AB in the ratio p : q, then the position vector of P is given by $\underline{r}$ = ----------
(a) $\frac{p\underline{a} - q\underline{b}}{p - q}$ (b) $\frac{p\underline{a} + q\underline{b}}{p + q}$
(c) $\frac{p\underline{a} - q\underline{b}}{p + q}$ (d) $\frac{p\underline{b} + q\underline{a}}{p + q}$

Q.25 The vector $\underline{u} = [x, y]$ in R^2 can be uniquely represented by
(a) $x\underline{i} + y\underline{j}$ (b) $x\underline{i} - y\underline{j}$
(c) $y\underline{i} + x\underline{j}$ (d) $y\underline{i} + x\underline{j}$

Q.26 The null or zero vector in R^3 is $\underline{O}$ = ----------.
(a) [0] (b) [0, 0]
(c) [0, 0, 0] (d) None of these

Q.27 If terminal point B of a vector $\overrightarrow{AB}$ coincides with its initial point A, then |AB| =
(a) 0 (b) 2
(c) 1 (d) None of these

Q.28 $\underline{O}$ = [0, 0] is called the
(a) Unit vector (b) Null vector
(c) Position vector (d) None of these

Q.29 The vector $\underline{j}$ = [0, 1] is called unit vectors along
(a) x-axis (b) y-axis
(c) z-axis (d) None of these

Q.30 Two vectors $\underline{u} = [x_1, y_1, z_1]$ and $\underline{v} = [x_2, y_2, z_2]$ are equal if and only if:
(a) $x_1 = x_2 = y_1 = y_2 = z_1 = z_2$
(b) $y_1 = y_2, z_1 = z_2$
(c) $x_1 = x_2, y_1 = y_2, z_1 = z_2$
(d) None of these

Q.31 Let $\underline{u}$ and $\underline{v}$ be any two vectors in plane or space have same initial point. Then the dot product of u and v is defined by $\underline{u}.\underline{v}$ = ---------, where θ is the angle between $\underline{u}$ and $\underline{v}$ and lies in [0, π].
(a) $\underline{u}.\underline{v} \cos\theta$ (b) $|\underline{u}|.|\underline{v}|$
(c) $|\underline{u}|.|\underline{v}| \cos\theta$ (d) None of these

Q.32 A unit vector is defined as a vector whose magnitude is: *(Multan Board 2017 G-I)*
(a) 0 (b) 2
(c) 1 (d) None of these

Q.33 A vector, whose initial point is the origin O and whose terminal point P, is called the ---------- of the point P and is written as $\overrightarrow{OP}$.
(a) Unit vector (b) Null vector
(c) Position vector (d) None of these

Q.34 A vector $\underline{u}$ is called a unit vector, if $|\underline{u}|$ = ---------
(a) 0 (b) 2
(c) 1 (d) −1

Q.35 A vector $\underline{v}$ is ---------- to a vector $k\underline{v}$, where k is a scalar.
(a) Equal (b) Perpendicular
(c) Parallel (d) None of these

Q.36 The vector $\underline{i} = [1, 0]$ is called --------- along x-axis.
(a) Unit vector (b) Null vector
(c) Position vector (d) None of these

Q.37 Two vectors are --------- if and only if they are non-zero scalar multiple of each other.
(a) Equal (b) Perpendicular
(c) Parallel (d) None of these

Q.38 The magnitude, length or norm of vector $\underline{u} = [x, y]$ is:
(a) $\sqrt{x^2 - y^2}$ (b) $x^2 + y^2$
(c) $\sqrt{y^2 - x^2}$ (d) $\sqrt{x^2 + y^2}$

Q.39 Which of the following is a vector quantity.
(a) Entropy (b) Weight
(c) Calorie (d) Kinetic energy

Q.40 Which of the following is a vector quantity.
(a) Centrifugal force (b) Time
(c) Temperature (d) Distance

Q.41 Which of the following is not a vector quantity.
(a) Weight (b) Mass
(c) Force (d) Velocity

Q.42 If $\left|\frac{\rightarrow}{a}\right| = 1$, then $\vec{a}$ is a
(a) Free vector (b) Unit vector
(c) Null vector (d) None of these

Q.43 Let $\vec{a}$ be a non-zero vector, then $\frac{\vec{a}}{|\vec{a}|}$ is a
(a) Scalar quantity
(b) Unit vector parallel to $\vec{a}$
(c) Unit vector perpendicular to $\vec{a}$
(d) Reciprocal vector

Q.44 If $\vec{a} = \lambda \hat{a}$, then λ is equal to:
(a) $\frac{\vec{a}}{|\vec{a}|}$ (b) $\frac{1}{|\vec{a}|}$
(c) $\frac{1}{2}|\vec{a}|$ (d) $|\vec{a}|$

Q.45 Mark the wrong statement. Two vectors can be
(a) Added (b) Subtracted
(c) Multiplied (d) Divided

Q.46 If P(x, y, z) be any point in space, then $\overrightarrow{OP} = x\,i + y\,j + z\,k$ is
(a) Position vector of O (d) Position vector of P
(c) Unit vector (d) Displacement

Q.47 The difference between the position vectors of two points is called the
(a) Distance (b) Velocity
(c) Speed (d) Displacement

Q.48 Two vectors are equal if they
(a) Passes through the same point
(b) Are parallel to each other
(c) Area parallel to each other and have same direction
(d) Have equal magnitude and have same direction

Q.49 If terminal point B of a vector $\overrightarrow{AB}$ coincides with its initial point A, then $\overrightarrow{AB}$ is known as
(a) Scalar (b) Free vector
(c) Unit vector (d) Null vector

Q.50 If $\underline{u} = [x, y]$ and $\underline{v} = [x', y']$ are two vectors, then difference between two vectors are $\underline{u} - \underline{v} =$
(a) $[x - y, x' - y']$ (b) $[x - y, x' - y']$
(c) $[x + x', y + y']$ (d) $[x - x', y - y']$

Q.51 The vector $\underline{i} = [1, 0]$ is called unit vectors along
(a) x-axis (b) z-axis
(c) y-axis (d) None of these

Q.52 The vector j = [0,1] is called ------- along y-axis.
(a) Position vector (b) Null vector
(c) Unit vector (d) None of these

Q.53 Two vectors are said to be negative of each other if they have ---------- magnitude but ---------- direction.
(a) Same, same (b) Opposite, same
(c) Same, opposite (d) None of these

Q.54 If $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, then $\overrightarrow{AB}$ = *(Gujranwala Board 2019 G-I)*
(a) $\vec{a} - \vec{b}$ (b) $\vec{a} + \vec{b}$
(c) $\vec{b} - \vec{a}$ (d) $\vec{b} + \vec{a}$

EXERCISE 7.2

SHORT ANSWERS TO THE QUESTIONS

Q.1 If $\bar{v} = 3\underline{i} - 2\underline{j} + 2\underline{k}$, $\bar{w} = 5\underline{i} - \underline{j} + 3\underline{k}$ find $|3\bar{v} + \bar{w}|$.
(Faisalabad Board 2013) (Rawalpindi Board 2013)

Ans. $3\bar{v} + \bar{w} = 3(3\underline{i} - 2\underline{j} + 2\underline{k}) + 5\underline{i} - \underline{j} + 3\underline{k}$

$3\bar{v} + \bar{w} = 9\underline{i} - 6\underline{j} + 6\underline{k} + 5\underline{i} - \underline{j} + 3\underline{k}$

$3\bar{v} + \bar{w} = 14\underline{i} - 7\underline{j} + 9\underline{k}$

$|3\bar{v} + \bar{w}| = \sqrt{(14)^2 + (-7)^2 + (9)^2}$

$|3\bar{v} + \bar{w}| = \sqrt{196 + 49 + 81}$

$|3\bar{v} + \bar{w}| = \sqrt{326}$

Q.2 Find α so that $|a\hat{i} + (\alpha + 1)\hat{j} + 2\hat{k}| = 3$.
(Lahore Board 2013)(Rawalpindi Board 2017 G-II)
(Multan Board 2013, 2018 G-II, 2017 G-I)
(Rawalpindi Board 2016)(Lahore Board 2017 G-I)
(Bahawalpur, Sargodha Board 2018)

Ans. $|a\hat{i} + (\alpha + 1)\hat{j} + 2\hat{k}| = 3$

$\sqrt{\alpha^2 + (\alpha + 1)^2 + 4} = 3$

Squaring

$\alpha^2 + \alpha^2 + 2\alpha + 1 + 4 = 9$

$2\alpha^2 + 2\alpha + 5 - 9 = 0$

$2\alpha^2 + 2\alpha - 4 = 0$

$\alpha^2 + a - 2 = 0$

$\alpha^2 + 2\alpha - \alpha - 2 = 0$

$\alpha(\alpha + 2) - 1(\alpha + 2) = 0$

$(\alpha + 2)(\alpha - 1) = 0$

$\Rightarrow \alpha = 1, -2$

Q.3 Find a vector whose magnitude is 4 and is parallel to $2\underline{i} - 3\underline{j} + 6\underline{k}$
(Lahore Board 2016, 2018 G-I)
(Bahawalpur Board 2016)(Rawalpindi Board 2017 G-II)
(Gujranwala Board 2019 G-I)

Ans. Let $\underline{a} = 2i - 3j + 6k$

$|\underline{a}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$

$\therefore$ Unit vector of $\underline{a} = \hat{a} = \dfrac{\underline{a}}{|\underline{a}|} = \dfrac{1}{7}(2i - 3j + 6k)$

Thus, required vector parallel to $\underline{a}$ is

$4\hat{a} = 4\left[\dfrac{1}{7}(2\underline{i} - 3\underline{j} + 6\underline{k})\right] = \dfrac{8}{7}\underline{i} - \dfrac{12}{7}\underline{j} + \dfrac{24}{7}\underline{k}$

Q.4 Find a vector of length 5 in the opposite direction of $v = i - 2j + 3k$.
(Gujranwala Board 2013, 2019 G-II)
(Multan Board 2013 G-I)(Lahore Board 2017 G-II)

Ans. $v = i - 2j + 3k$

$|v| = \sqrt{(1)^2 + (-2)^2 + (3)^2}$

$|v| = \sqrt{1 + 4 + 9} = \sqrt{14}$

$\vec{v} = \frac{v}{|v|} = \frac{i - 2j + 3k}{\sqrt{14}}$

Required vector $= 5(\vec{v}) = \frac{-5(i - 2j + 3k)}{\sqrt{14}}$

$= \frac{-5}{\sqrt{14}}(i - 2j + 3k)$

Q.5 Find the constant a so that the vectors $v = i - 3i + 4k$ and $w = ai + 9j - 12k$ are parallel. *(Multan Board 2014 G-II)*

Ans. Given $v = i - 3i + 4k$ and $w = ai + 9j - 12k$

Since v and w are parallel.

So $\frac{1}{a} = \frac{-3}{9} = \frac{4}{-12} \Rightarrow \frac{1}{a} = \frac{-1}{3} \Rightarrow a = -3$

Q.6 Find the direction cosines of the vector $6i - 2j + k$. *(Sargodha Board 2010, 2013)*
(Gujranwala Board 2010, 2017)

Ans. Let $v = 6i - 2j + k$

$|v| = \sqrt{(6)^2 + (-2)^2 + (1)^2}$

$|v| = \sqrt{36 + 4 + 1}$

$|v| = \sqrt{41}$

$\bar{v} = \frac{v}{|v|} = \frac{6i - 2 + k}{\sqrt{41}}$

So, the direction cosines are $\frac{6}{\sqrt{41}}, \frac{-2}{\sqrt{41}}, \frac{1}{\sqrt{41}}$

Q.7 Find the direction cosines of the vector $\overrightarrow{PQ}$, where $P(2,1,5)$ and $Q(1,3,1)$
(Lahore Board 2015 G-I)(Multan Board 2018 G-II)
(Sahiwal Board 2019)

Ans. $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (i + 3j + k) - (2i + j + 5k)$

$= -i + 2j - 4k$

$\left|\overrightarrow{PQ}\right| = \sqrt{(-1)^2 + (2)^2 + (-4)^2} = \sqrt{1 + 4 + 16} = \sqrt{21}$

The direction cosines of $\overrightarrow{PQ}$ are $\frac{-1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}$

Q.8 Find the direction cosine for the vector $v = 3i - j + 2k$ *(D.G Khan Board 2017 G-I)*
(A.J.K Board 2017)(Lahore Board 2015 G-II)

Ans.

$|v| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}$

Direction cosines are $\frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}$

Q.9 Find the direction cosine of vector $v = i - j - k$ *(Multan Board 2018 G-I)*

Ans. $v = i - j - k$

$|v| = \sqrt{1^2 + (-1)^2 + (-1)^2}$

$= \sqrt{3}$

Direction cosines of v are

$\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

Q.10 Find a unit vector in the direction of $\bar{v} = i + 2j - k$ *(Faisalabad Board 2019)*

Ans. Let $v = i + 2j - k$

$|v| = \sqrt{(1)^2 + (2)^2 + (-1)^2}$

$= \sqrt{1 + 4 + 1} = \sqrt{6}$

$\hat{v} = \frac{v}{|v|} = \frac{i + 2j - k}{\sqrt{6}} = \frac{1}{\sqrt{6}}i + \frac{2}{\sqrt{6}}j - \frac{1}{\sqrt{6}}k$

LONG QUESTIONS

Q.1 Find a vector whose magnitude is 4 and parallel to $2i - 3j + 6k$. *(Sargodha Board 2013)*

Sol: See Short Question 9

Q.2 Find two vectors of length 2 parallel to the vector. $v = 2i - 4j + 4k$. *(Gujranwala Board 2014)*
(D.G Khan Board 2017 G-II)
(Rawalpindi Board 2017 G-II)

Sol: Let $v = 2i - 4j + 4k$

$|v| = \sqrt{(2)^2 + (-4)^2 + (4)^2} = \sqrt{4 + 16 + 16}$

$= \sqrt{36} = 6$

$\hat{v} = \frac{2i - 4j + 4k}{6}$

required vectors $= \pm 2\hat{v} = \pm 2 \times \frac{2i - 4j + 4k}{6}$

$= \pm \frac{1}{3}(2i - 4j + 4k)$

So required vectors are

$\frac{2}{3}i - \frac{4}{3}j + \frac{4}{3}k$ (same direction) and

$-\frac{2}{3}i + \frac{4}{3}j - \frac{4}{3}k$ (opposite direction)

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 If α, β, γ are direction angles of a vector, then $Cos^2\alpha + Cos^2\beta + Cos^2\gamma =$ *(D.G.K Board 2014, G-II)*

(a) 0 (b) 1
(c) 2 (d) 3

Q.2 Projection of $\vec{b}$ along $\vec{a}$ is: *(D.G.K Board 2013 G-II)*

(a) $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$ (b) $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}\ \vec{b}|}$
(c) $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ (d) $\frac{\vec{b} \cdot \vec{a}}{|\vec{b}\ \vec{a}|}$

Q.3 If $3\underline{i} + \underline{j} - \underline{k}$, and $\lambda\underline{i} - 4\underline{j} + 4\underline{k}$ are parallel vectors, then value of λ is: *(D.G.K Board 2012 G-I)*

(a) 3 (b) −3
(c) −12 (d) −4

Q.4 If $\underline{i} - 3\underline{j} + 4\underline{k}$ and $\lambda\underline{i} + 9\underline{j} - 12\underline{k}$ are parallel, then λ equals:

(a) −3 (b) 3
(c) −2 (d) 2

Q.5 Two vectors $\underline{a}$ and $\underline{b}$ are co-planar if $p\vec{a} + q\vec{b} = 0$ implies: *(Multan Board 2012 G-I)*

(a) $p = 0, q \neq 0$ (b) $p \neq 0, q = 0$
(c) $p = 0, q = 0$ (d) $p \neq 0, q \neq 0$

Q.6 Direction cosine of z-axis are. *(Gujranwala Board 2018)*

(a) [1, 0, 0] (b) [1, 1, 1]
(c) [0, 1, 0] (d) [0, 0, 1]

Q.7 Magnitude of vector $2\underline{i} + 3\underline{j} + 4\underline{k}$ is: *(Lahore Board 2014 G-II)*

(a) 29 (b) $\sqrt{29}$
(c) 28 (d) $\sqrt{28}$

Q.8 $|Cos\ \alpha\ \underline{i} + Sin\ \alpha\underline{j} + \alpha\underline{k}| =$ *(Lahore Board 2012 G-I)*

(a) 0 (b) 1
(c) −1 (d) 2

Q.9 Which of triples can be direction angles of a single vector = *(Rawalpindi Board 2016)*

(a) 90°, 90°, 45° (b) 0°, 0°, 45°
(c) 45°, 45°, 90° (d) 30°, 30°, 30°

Q.10 If α, β, γ be the direction angles of a vector, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$ equals. *(Lahore Board 2016 G-I)*

(a) −1 (b) 0
(c) 1 (d) 2

Q.11 The vector $\underline{i} = [1, 0, 0]$ is called -------- along x-axis.

(a) Unit vector (b) Null vector
(c) Position vector (d) None of these

Q.12 The vector $\underline{j} = [0, 1, 0]$ is called --------- along y-axis.

(a) Unit vector (b) Null vector
(c) Position vector (d) None of these

Q.13 The vector $\underline{k} = [0, 0, 1]$ is called ---------- along z-axis.

(a) Unit vector (b) Null vector
(c) Free vector (d) None of these

Q.14 The sum of squares of direction angles is equal to:

(a) 0 (b) Any vector
(c) 1 (d) None of these

Q.15 The vector $\underline{j} = [0, 1, 0]$ is called unit vector along

(a) x-axis (b) y-axis
(c) z-axis (d) None of these

Q.16 The vector $\underline{k} = [0, 0, 1]$ is called unit vector along

(a) x-axis (b) y-axis
(c) z-axis (d) None of these

Q.17 The vector $\underline{i} = [1, 0, 0]$ is called unit vector along

(a) x-axis (b) y-axis
(c) z-axis (d) None of these

Q.18 The magnitude, length or norm of vector $\underline{u} = [x, y, z]$ is $|\underline{u}| =$ ----------

(a) $\sqrt{x^2 - y^2 - z^2}$ (b) $\sqrt{x^2 + y^2}$
(c) $x^2 + y^2 + z^2$ (d) $\sqrt{x^2 + y^2 + z^2}$

Q.19 $Cos^2\alpha + \cos^2\beta + \cos^2\gamma =$ --------.

(a) 0 (b) 2
(c) 1 (d) 3

Q.20 The magnitude of the vector $\vec{v} = a\underline{i} + b\underline{j} + c\underline{k}$. *(A.J.K Board 2017)*

(a) $a^2 + b^2 + c^2$ (b) $a + b + c$
(c) $a^3 + b^3 + c^3$ (d) $\sqrt{a^2 + b^2 + c^2}$

Q.21 If $\vec{V} = -\underline{i} - 2\underline{j} - 3\underline{k}$, then $|\underline{v}| =$ *(D.G.K Board 2012 G-II)*

(a) $-\sqrt{6}$ (b) −14
(c) $\sqrt{14}$ (d) 6

Q.22 Length of vector $2\underline{i} - \underline{j} - 2\underline{k}$ is. *(Lahore Board 2018 G-I)*

(a) 2 (b) 4
(c) 3 (d) 5

Q.23 The direction cosines of y-axis are.
(Lahore Board 2018 G-I)
(a) [0, 1, 0] (b) [1, 0, 0]
(c) [0, 0, 1] (d) [0, 0, 0]

Q.24 If $\vec{V}$ is any vector then vector of magnitude 5 opposite to $\vec{V}$ is. *(Lahore Board 2018 G-I)*
(a) $5\vec{V}$ (b) $-5\vec{V}$
(c) $5\dfrac{\vec{V}}{|\vec{V}|}$ (d) $-5\dfrac{\vec{V}}{|\vec{V}|}$

Q.25 For a vector $v = 2i + 3j - 6k$, $\cos\beta =$
(Multan Board 2018 G-II)
(a) $-\dfrac{6}{7}$ (b) $\dfrac{2}{7}$
(c) $\dfrac{3}{7}$ (d) $\dfrac{-3}{7}$

Q.26 If α is the direction angle of a vector, then
(Sargodha Board 2018)
(a) $0 < \alpha < \pi$ (b) $0 \le \alpha \le \pi$
(c) $0 < \alpha \le \pi$ (d) $0 \le \alpha < \pi$

Q.27 The position vector of any point in xy-plane is:
(Faisalabad Board 2019 G-II)
(a) $x\,i + y\,j + z\,k$ (b) $y\,j + z\,k$
(c) $x\,i + y\,j$ (d) $x\,i + z\,k$

Q.28 $|\cos\alpha\, i + \sin a\, j + 0\, k| =$
(Gujranwala Board 2019 G-I)
(a) 0 (b) -1
(c) 2 (d) 1

Q.29 If α, β, γ be the direction angles of a vector then $\cos^2\alpha + \cos^2\beta + \cos\gamma$
(Gujranwala Board 2019 G-II)
(a) 2 (b) 0
(c) -1 (d) 1

Q.30 Magnitude of vector $6i + 3j - 2k$ is:
(Multan Board 2019)
(a) 7 (b) 6
(c) 3 (d) -2

Q.31 Direction cosines of y – axis are:
(Multan Board 2019)
(a) 0, 0, 1 (b) 1, 0, 0
(c) 0, 1, 0 (d) $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

EXERCISE 7.3

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find α so that vectors
$u = \alpha i + 2\alpha j - k$, $v = i + \alpha j + 3k$
are perpendicular. *(A.J.K Board 2017)*
(Lahore Board 2015 G-III)(Gujranwala Board 2011)
(Bahawalpur Board 2016, 2019)
(Sargodha Board 2017)(Multan Board 2018 G-I)

Ans. Since u and v are perpendicular, so $u \cdot v = 0$

$$(\alpha i + 2\alpha j - k)\cdot(i + \alpha j + 3k) = 0$$
$$(\alpha)(1) + (2\alpha)(\alpha) + (-1)(3) = 0$$
$$\alpha + 2\alpha^2 - 3 = 0 \Rightarrow 2\alpha^2 + \alpha - 3 = 0$$
$$\Rightarrow 2\alpha^2 + 3\alpha - 2\alpha - 3 = 0$$
$$\Rightarrow \alpha(2\alpha + 3) - 1(2\alpha + 3) = 0$$
$$\Rightarrow (2\alpha + 3)(\alpha - 1) = 0$$
$$\Rightarrow 2\alpha + 3 = 0 \text{ or } \alpha - 1 = 0 \Rightarrow \alpha = \frac{-3}{2} \text{ or } \alpha = 1$$

LONG QUESTIONS

Q.1 Find x so that points A(1, –1, 0), B(–2, 2, 1) and C(0, 2, x) form triangle with right angle at c.

Sol: $\vec{AC} = (0-1)i + (2+1)j + (x-0)k$
$= -i + 3j + xk$

$\vec{BC} = (0+2)i + (2-2)j + (x-1)k$
$= 2i + 0j + (x-1)k$

Since there is right angle at c then $\vec{AC}\cdot\vec{BC} = 0$
$(-i + 3j + xk)\cdot(2i + 0j + (x-1)k) = 0$
$(-1)(2) + 3(0) + x(x-1) = 0$
$-2 + 0 + x^2 - x = 0$
$x^2 - x - 2 = 0$
$(x-2)(x+1) = 0$

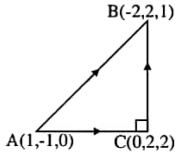
Either $x - 2 = 0$, $x = 2$ | Or $x + 1 = 0$, $x = -1$

Q.2 Find the number Z, so that the triangle with vertices A(1, –1, 0), B(–2, 2, 1) and C(0, 2, Z) is a right angle triangle with right angle at C.
(Rawalpindi Board 2016)

Sol:

$\vec{AC} = (0-1)i + (2+1)j + (z-0)k$
$= -i + 3j + zk$

$\vec{CB} = (-2-0)i + (2-2)j + (1-z)k$
$= -2i + 0j + (1-z)k$



Since right angle is at C so $\overrightarrow{AC} \perp \overrightarrow{CB}$

$\Rightarrow \quad \overrightarrow{AC} \bullet \overrightarrow{CB} = 0$

$(-i + 3\underline{j} + z\underline{k}) \bullet (-2\underline{i} + 0\underline{j} + (1 - z)\underline{k}) = 0$

$(-1)(-2) + (3)(0) + z(1 - z) = 0$

$2 + 0 + z - z^2 = 0$

$\Rightarrow \quad z^2 - z - 2 = 0$

$z^2 - 2z + z - 2 = 0$

$z(z - 2) + 1(z - 2) = 0$

$(z - 2)(z + 1) = 0$

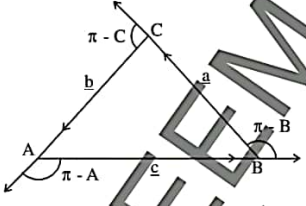
$\Rightarrow \quad z - 2 = 0 \ , z + 1 = 0$

$\Rightarrow \quad z = 2, \quad z = -1$

Q.3 Prove vectorically that in any triangle ABC, $a^2 = b^2 + c^2 - 2bc \cos A$.

(Lahore Board 2016 G-I)(Gujranwala Board 2016) (Multan Board 2018 G-I)

Sol: $a^2 = b^2 + c^2 - 2bc \cos A$



Let the vectors $\underline{a}, \underline{b}, \underline{c}$ be along the sides $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ of the triangle ABC respectively.

$\therefore \quad \underline{a} + \underline{b} + \underline{c} = 0$

$\Rightarrow \quad \underline{a} = -(\underline{b} + \underline{c})$

now $\underline{a} \bullet \underline{a} = (\underline{b} + \underline{c}) \bullet (\underline{b} + \underline{c})$

$\underline{a}.\underline{a} = \underline{b}.\underline{b} + \underline{b}.\underline{c} + \underline{b}.\underline{a} + \underline{c}.\underline{c}$

$a^2 = b^2 + 2|\underline{b}|\,|\underline{c}| \cos(\pi - A) + c^2$

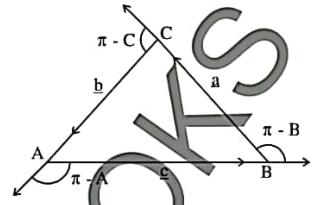
$\Theta \cos(\pi - \theta) = -\cos\theta$

$a^2 = b^2 + 2bc(-\cos A) + c^2$

$a^2 = b^2 + c^2 - 2bc \cos A$

Q.4 Prove that in any triangle ABC $b^2 = c^2 + a^2 - 2ac \cos\beta$. *(Sargodha Board 2016)*

Sol:



Let the vectors $\underline{a}, \underline{b}, \underline{c}$ be along the sides $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ of the triangle ABC respectively.

$\therefore \quad \underline{a} + \underline{b} + \underline{c} = 0$

$\Rightarrow \quad \underline{b} = -(\underline{a} + \underline{c})$

now $\underline{b} \bullet \underline{b} = (\underline{a} + \underline{c}) \bullet (\underline{a} + \underline{c})$

$\underline{b}.\underline{b} = \underline{a}.\underline{a} + \underline{a}.\underline{c} + \underline{c}.\underline{a} + \underline{c}.\underline{c}$

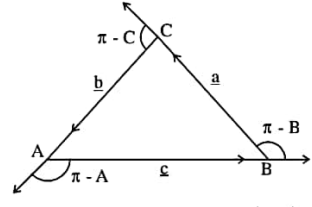
$b^2 = a^2 + c^2 + 2\overrightarrow{a} \,.\overrightarrow{c}$

$b^2 = a^2 + c^2 + 2ac \cos(\pi - B)$

$b^2 = a^2 + c^2 - 2ac \cos B$

Q.5 Prove that in a triangle ABC, $c^2 = a^2 + b^2 - 2ab \cos C$. *(Faisalabad Board 2017) (Rawalpindi Board 2013)(Faisalabad Board 2016)*

Sol:



Let the vectors $\underline{a}, \underline{b}, \underline{c}$ be along the sides $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ of the triangle ABC respectively.

$\therefore \quad \underline{a} + \underline{b} + \underline{c} = 0$

$\Rightarrow \quad \underline{c} = -(\underline{a} + \underline{b})$

now $\underline{c} \bullet \underline{c} = (\underline{a} + \underline{b}) \bullet (\underline{a} + \underline{b})$

$\underline{c}.\underline{c} = \underline{a}.\underline{a} + \underline{a}.\underline{b} + \underline{b}.\underline{a} + \underline{b}.\underline{b}$

$c^2 = a^2 + 2|\underline{a}||\underline{b}| \cos(\pi - C) + b^2$

$c^2 = a^2 + 2ab(-\cos C) + b^2$

$\Theta \cos(\pi - \theta) = -\cos\theta$

$c^2 = a^2 + b^2 - 2ab \cos C$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 If $\vec{a}$ and $\vec{b}$ are oppositely directed then $\vec{a} \cdot \vec{b}$ equals: *(Multan Board 2013 G-I)*

(a) ab (b) −ab
(c) ab Sinθ (d) ab Cos θ

Q.2 If the vectors $2\underline{i} + 4\underline{j} - 7\underline{k}$ and $2\underline{i} + 6\underline{j} + x\underline{k}$ are perpendicular then x =
(D.G.K Board 2013-14 G-I, Multan Board 2014-15 GII) (Gujranwala Board 2013)

(a) 8 (b) 2
(c) 1 (d) 4

Q.3 $\hat{i} \cdot \hat{k} = ?$ *(D.G.K Board 2014, G-I)*

(a) 1 (b) 0
(c) −1 (d) 2

Q.4 Projection of $\vec{a} = \underline{i} - \underline{k}$ along $\vec{b} = \underline{j} + \underline{k}$

(a) $\frac{-1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$
(c) $\frac{3}{\sqrt{2}}$ (d) $\frac{1}{2}$

Q.5 Projection of $\underline{V}$ along $\underline{U}$ is:
(D.G.K Board 2012 G-I)(Sahiwal Board 2018)

(a) $\underline{U}.\hat{V}$ (b) $\underline{V}.\hat{U}$
(c) $\frac{\underline{U} \cdot \underline{V}}{|\underline{V}|}$ (d) $\frac{\underline{U} \times \underline{V}}{|\underline{V}|}$

Q.6 Dot product of two vectors is: *(D.G.K Board 2010)*

(a) Associative (b) Commutative
(c) Not defined (d) Imaginary

Q.7 $\underline{i} \cdot \underline{i}$ = *(Multan Board 2015 G-II)*

(a) 0 (b) 1
(c) 2 (d) 3

Q.8 Two vectors $\underline{a}$ and $\underline{b}$ are perpendicular if:
(Multan Board 2012 G-II)

(a) $\underline{a} + \underline{b} = 0$ (b) $\underline{a} - \underline{b} = 0$
(c) $\underline{a} \cdot \underline{b} = 0$ (d) $\underline{a} \cdot \underline{b} = 1$

Q.9 For two non-zero vectors $\underline{a}$ and $\underline{b}$, $\underline{a}.\underline{b}$ equals:
(Multan Board 2011 G-I)

(a) ab cosθ (b) ab sinθ
(c) ab (d) −ab

Q.10 Projection of vector $\underline{U} = a\underline{i} + b\underline{j} + ck$ along $\underline{i}$ is:
(Lahore Board 2013)

(a) c (b) a
(c) b (d) 0

Q.11 If $\underline{U} = 2\alpha\ \underline{i} + \underline{j} - \underline{k}$, $\underline{V} = \underline{i} + \alpha\underline{j} + 4\underline{k}$ are perpendicular, the a *(Bahawalpur Board 2016)*

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$
(c) $\frac{4}{3}$ (d) 0

Q.12 For any two vector $\vec{a}$ antl $\vec{b}$ projection of $\vec{a}$ on $\vec{b}$ is:
(Faisalabad Board 2016)

(a) $\frac{\vec{a}.\vec{b}}{a}$ (b) $\frac{a.b}{b}$
(c) $\frac{a.b}{ab}$ (d) $\frac{a.b}{}$

Q.13 For a vector $\vec{v} = a\underline{i} + b\underline{j} + c\underline{k}$, projection of $\vec{v}$ along $\underline{k}$ is: *(Sargodha Board 2016)*

(a) a (b) b
(c) c (d) a + b + c

Q.14 The dot product of unit vector $\underline{i}$ with unit vector $\underline{i}$ is:

(a) 0 (b) 2
(c) 1 (d) 3

Q.15 The dot product of unit vector $\underline{j}$ with unit vector $\underline{j}$ is:

(a) 0 (b) 2
(c) 1 (d) 3

Q.16 If $|\vec{a}| = |\vec{b}|$, then $(\vec{a} + \vec{b}).(\vec{a} - \vec{b})$ equals

(a) 0 (b) 1
(c) 2 (d) -1

Q.17 If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, then $\vec{a}$ and $\vec{b}$ are

(a) Parallel
(b) Perpendicular
(c) Angle between $\vec{a}$ and $\vec{b}$ is 45°
(d) Angle between $\vec{a}$ and $\vec{b}$ is 60°

Q.18 The dot product of unit vector $\underline{k}$ with unit vector $\underline{k}$ is:

(a) 0 (b) 2
(c) 1 (d) None of these

Q.19 Two non-zero vectors $\underline{u}$ and $\underline{v}$ are perpendicular if and only if $\underline{u}.\underline{v}$ is equal to:

(a) 0 (b) 2
(c) 1 (d) None of these

Q.20 If θ = ---------- between two vectors $\underline{u}$ and $\underline{v}$, then u and v are perpendicular to each other.

(a) 0 (b) $\frac{\pi}{2}$
(c) π (d) $\frac{\pi}{4}$

Q.21 If θ is the angle between two vectors $\underline{u}$ and $\underline{v}$, then θ =

(a) $\text{Cos}\left\{\frac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|}\right\}$ (b) $\text{Cos}^{-1}(\underline{u} \cdot \underline{v})$
(c) $\text{Cos}^{-1}\left\{\frac{1}{|\underline{u}||\underline{v}|}\right\}$ (d) $\text{Cos}^{-1}\left\{\frac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|}\right\}$

Q.22 If $\underline{u} = [x_1, y_1]$ and $\underline{v} = [x_2, y_2]$, are any two non-zero vectors in the plane, then dot product of $\underline{u}$ and $\underline{v}$ is $\underline{u} \cdot \underline{v}$ = ----------
(a) $x_1x_2 - y_1y_2$ (b) $x_1x_2 + y_1y_2$
(c) $x_1y_2 + x_2y_1$ (d) $x_1y_2 - x_2y_1$

Q.23 The dot product of unit vector $\underline{i}$ with unit vector $\underline{k}$ is:
(a) 0 (b) 2
(c) 1 (d) 3

Q.24 The dot product of unit vector $\underline{j}$ with unit vector $\underline{k}$ is:
(a) 0 (b) 2
(c) 1 (d) 3

Q.25 u . v = ----------, where u and v are any vector.
(a) $-\underline{v} \cdot \underline{u}$ (b) $\underline{v} \cdot \underline{u}$
(c) $-\underline{u} \cdot \underline{v}$ (d) None of these

Q.26 The projection of $\underline{v}$ along $\underline{u}$ is equal to:
(a) $\frac{\underline{u} \cdot \underline{v}}{|\underline{u}|}$ (b) $\frac{\underline{u} \cdot \underline{v}}{|\underline{v}|}$
(c) $\underline{u} \cdot \underline{v}$ (d) None of these

Q.27 If θ = ---------- between two vectors $\underline{u}$ and $\underline{v}$, then $\underline{u}$ and $\underline{v}$ are parallel vectors.
(a) π (b) $\frac{\pi}{2}$
(c) $\frac{3\pi}{2}$ (d) $\frac{\pi}{4}$

Q.28 The dot product of unit vector $\underline{j}$ with unit vector $\underline{i}$ is:
(a) 0 (b) 2
(c) 1 (d) 3

Q.29 The dot product of unit vector $\underline{k}$ with unit vector $\underline{j}$ is:
(a) 0 (b) 2
(c) 1 (d) 3

Q.30 If θ = --------- between two vectors $\underline{u}$ and $\underline{v}$, then $\underline{u}$ and $\underline{v}$ are collinear.
(a) 0 (b) 1
(c) $\frac{3\pi}{2}$ (d) $\frac{\pi}{2}$

Q.31 The dot product of unit vector $\underline{k}$ with unit vector $\underline{i}$ is:
(a) 0 (b) 2
(c) 1 (d) 3

Q.32 If $\underline{u} = [x_1, y_1, z_1]$ and $\underline{v} = [x_2, y_2, z_2]$ are any three non-zero vectors, then $\underline{u} \cdot \underline{v}$ =
(a) $x_1x_2 + y_1z_2 + z_1y_2$ (b) $x_1z_2 + y_1y_2 + z_1x_2$
(c) $x_1y_1 + x_2y_2 + z_1z_2$ (d) $x_1x_2 + y_1y_2 + z_1z_2$

Q.33 The dot product of unit vector $\underline{j}$ with unit vector $\underline{j}$ is:
(a) 0 (b) $\underline{i}$
(c) 1 (d) $\underline{j}$

Q.34 Cosine of the angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is. *(Faisalabad Board 2017)*
(a) $\underline{a} \cdot \underline{b}$ (b) $\frac{|\underline{a}|\,|\underline{b}|}{\vec{a} \cdot \vec{b}}$
(c) $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$ (d) $\frac{\vec{a} \times \vec{b}}{|\vec{a}|\,|\vec{b}|}$

Q.35 If vectors $2\underline{i} + \underline{j} + \underline{k}$ and $\underline{i} - 4\underline{j} + \alpha\underline{k}$ are perpendicular then α = *(Lahore Board 2017 G-II)*
(a) 1 (b) 2
(c) 3 (d) 4

Q.36 The projection of $-2\underline{i} + 3\underline{j} + 7\underline{k}$ on $2\underline{j} + \underline{k}$ is. *(Multan Board 2017 G-I)*
(a) $\frac{13}{5}$ (b) $\frac{13}{4}$
(c) $\frac{13}{\sqrt{5}}$ (d) 13

Q.37 $\underline{i} \cdot \underline{j}$ is equal to. *(Sahiwal Board 2017)*
(a) $\underline{j}$ (b) $\underline{i}$
(c) 1 (d) 0

Q.38 Projection of $\underline{u}$ along $\underline{v}$ is.
(a) $\frac{\underline{u}.\underline{v}}{|\underline{v}|}$ (b) $\frac{\underline{u}.\underline{v}}{|\underline{u}|}$
(c) $\frac{\underline{u}.\underline{v}}{|\underline{v}|\,|\underline{u}|}$ (d) $\underline{u} \cdot \underline{v}$

Q.39 The vectors $\underline{u}$ and $\underline{v}$ are parallel if angle between them is. *(Sargodha Board 2017)*
(a) $\frac{\pi}{2}$ (b) π
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

Q.40 An angle in a semi-circle is. *(Lahore Board 2017 G-I)*
(a) Right angle (b) Obtuse angle
(c) Reflexive angle (d) 0°

Q.41 Angle between the vectors $2\underline{i} + 3\underline{j} + \underline{k}$ and $2\underline{i} - \underline{j} - \underline{k}$ is. *(A.J.K Board 2017)*
(a) 30° (b) 45°
(c) 60° (d) 90°

Q.42 If $\underline{a} = \underline{i} - 2\underline{j} + 3\underline{k}$ and $\underline{b} = 3\underline{j} + 2\underline{k}$ are vectors, than angle between them is: *(Multan Board 2011 G-II)*
(a) 0 (b) $\frac{2}{\pi}$
(c) $\frac{\pi}{2}$ (d) π

Q.43 Projection of vector $\vec{U}$ on vector $\vec{V}$ is.

(Lahore Board 2018 G-II)

(a) $\frac{\vec{U}\cdot\vec{V}}{|\vec{V}|}$ (b) $\frac{\vec{U}\cdot\vec{V}}{|\vec{U}|}$

(c) $\frac{\vec{U}\times\vec{V}}{|\vec{V}|}$ (d) $\frac{\vec{U}\times\vec{V}}{|\vec{U}|}$

Q.44 cos θ is equal to. *(Rawalpindi Board 2018)*

(a) $\hat{a}\cdot\hat{b}$ (b) $|\vec{a}\times\vec{b}|$

(c) $\hat{a}\times\hat{b}$ (d) $\frac{|\vec{a}\times\vec{b}|}{|\vec{a}|}$

Q.45 Projection of $\vec{u} = a\,\underline{i} + b\,\underline{j} + c\,\underline{k}$ along $\underline{i}$ is:

(Bahawalpur Board 2019)

(a) b (b) a

(c) c (d) a + b

Q.46 Angle between the vectors $4\,\underline{i} + 2\underline{j} - \underline{k}$ and $-\,\underline{i} + \underline{j} - 2\underline{k}$ is *(Gujranwala Board 2019 G-II)*

(a) 30° (b) 45°

(c) 90° (d) 60°

Q.47 An angle in the semi circle is of measure:

(Lahore Board 2019)

(a) 30° (b) 60°

(c) 90° (d) 180°

Q.48 If $2\underline{i} + \alpha\,\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\,\underline{k}$ are perpendicular, then α =

(Rawalpindi Board 2019)

(a) 0 (b) 1

(c) −1 (d) 2

Q.49 It two vectors $\underline{i} - \underline{j} + \alpha\,\underline{k}$ and $\underline{i} - 2\underline{j} - 3\underline{k}$ are perpendicular, then 'α' will be equal to:

(Sargodha Board 2019)

(a) −2 (b) −3

(c) −1 (d) 1

Q.50 Two vectors $\underline{u}$ and $\underline{v}$ are perpendicular if

(Sahiwal Board 2019)

(a) $\underline{u}\cdot\underline{v} = 0$ (b) $\underline{u}\times\underline{v} = 0$

(c) $\underline{u} = t\,\underline{v}$ (d) $\underline{u} + \underline{v} = 0$

EXERCISE 7.4

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find $\underline{a}\times\underline{b}$ when $\underline{a} = [2, 1, 1]$ and $\underline{b} = [1, -1, 1]$.

(Faisalabad Board 2016)

Ans. $\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

$= \underline{i}(1-1) - \underline{j}(2+1) + \underline{k}(-2-1) = 0\underline{i} - 3\underline{j} - 3\underline{k}$

Q.2 If $\vec{a} = -4\underline{i} + \underline{j} - 2\underline{k}$, $\underline{b} = 2\underline{i} + \underline{j} + \underline{k}$ Find $\vec{a}\times\vec{b}$

(D.G Khan Board 2017 G-II)

Ans.

$\vec{a}\times\vec{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix}$

$= \underline{i}(1+2) - \underline{j}(-4+4) + \underline{k}(-4-2) = 3\underline{i} - 0\underline{j} - 6\underline{k}$

Q.3 Find a unit vector perpendicular to $\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}$ and $\underline{b} = 4\underline{i} + 3\underline{j} - \underline{k}$

(Rawalpindi Board 2016)(Sargodha Board 2017)

Ans. See Long Question 3

Q.4 If $\underline{a}\times\underline{b} = 0$ and $\underline{a}.\underline{b} = 0$, what conclusion can be drawn about $\underline{a}$ or $\underline{b}$? *(Gujranwala Board 2013)*

Ans. $\underline{a}\times\underline{b} = 0 \Rightarrow$ (i) $\underline{a}$, $\underline{b}$ are parallel or (ii) $\underline{a} = 0$ or $\underline{b} = 0$

$\underline{a}.\underline{b} = 0 \Rightarrow \underline{a}$, $\underline{b}$ are $\perp$ or (ii) $\underline{a} = 0$ or $b = 0$

So, we conclude that.

Either $\underline{a} = 0$ or $\underline{b} = 0$

Q.5 If $\underline{a}+\underline{b}+\underline{c} = 0$, then prove that $\underline{a}\times\underline{b} = \underline{b}\times\underline{c} = \underline{c}\times\underline{a}$.

(Lahore Board 2012, 2017 G-II)
(Bahawalpur Board 2014)(Sahiwal Board 2013)
(Faiusalabad Board 2017, 2019 G-I)

Ans. As $\underline{a}+\underline{b}+\underline{c} = 0$

$\underline{a}+\underline{b} = -\underline{c}$ Taking cross product with $\underline{b}$

$(\underline{a}+\underline{b})\times\underline{b} = -\underline{c}\times\underline{b}$

$\underline{a}\times\underline{b} + \underline{b}\times\underline{b} = -\underline{c}\times\underline{b}$

$\underline{a}\times\underline{b} + 0 = \underline{b}\times\underline{c}$ Q $\underline{b}\times\underline{c} = -\underline{c}\times\underline{b}$, $\underline{b}\times\underline{b} = 0$

$\underline{a}\times\underline{b} = \underline{b}\times\underline{c}$ .....(1)

Also taking cross product with $\underline{c}$

$\underline{c}\times(\underline{a}+\underline{b}) = -\underline{c}\times\underline{c}$

$\underline{c}\times\underline{a} + \underline{c}\times\underline{b} = 0$ Q $\underline{c}\times\underline{c} = 0$

$\underline{c}\times\underline{a} - \underline{b}\times\underline{c} = 0$ Q $\underline{c}\times\underline{b} = -\underline{b}\times\underline{c}$

$\underline{c}\times\underline{a} = \underline{b}\times\underline{c}$ ..... (2)

From (1) and (2), we have

$\underline{a}\times\underline{b} = \underline{b}\times\underline{c} = \underline{c}\times\underline{a}$

LONG QUESTIONS

Q.1 Find a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$ given below. Also find sine of angle between them where.

$\vec{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}$, $\vec{b} = 4\underline{i} + 3\underline{j} - \underline{k}$

(Bahawalpur Board 2016, 2019)

Sol:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix}$$

$= \underline{i}(6+9) - \underline{j}(-2+12) + \underline{k}(6+24) = 15\underline{i} - 10\underline{j} + 30\underline{k}$

$|\underline{a} \times \underline{b}| = \sqrt{(15)^2 + (-10)^2 + (30)^2}$

$= \sqrt{225 + 100 + 900} = \sqrt{1225} = 35$

Unit vector $= \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \dfrac{15\underline{i} - 10\underline{j} + 30\underline{k}}{35}$

$= \dfrac{3}{7}\underline{i} - \dfrac{2}{7}\underline{j} + \dfrac{6}{7}\underline{k}$

Now $|\underline{a}| = \sqrt{(2)^2 + (-6)^2 + (-3)^2} = \sqrt{4 + 36 + 9}$

$= \sqrt{49} = 7$

$|\underline{b}| = \sqrt{(4)^2 + (3)^2 + (-1)^2}$

$= \sqrt{16 + 9 + 1} = \sqrt{26}$

We know that

$$\sin\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{35}{7\sqrt{26}} = \frac{5}{\sqrt{26}}$$

Q.2 If $\underline{a} + \underline{b} + \underline{c} = 0$, then prove that $\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$.

(Lahore Board 2012 G-II)(Multan Board 2015 G-II) (Rawalpindi Board 2017 G-II)

Sol: See Short Question 14

Q.3 Prove that $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

(Lahore Board 2013 G-II)(Bahawalpur Board 2014) (Gujranwala Board 2014) (Sargodha Board 2017, 2019) (Multan Board 2015 G-I, 2016 G-II)

Sol: Let $\vec{OA}$ and $\vec{OB}$ be two unit vectors making angles α and β with positive x-axis such that

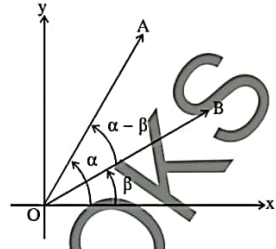
$m\angle BOA = \alpha - \beta$ and $|\vec{OA}| = |\vec{OB}| = 1$

Now

$\vec{OA} = \cos\alpha\underline{i} + \sin\alpha\underline{j}$

$\vec{OB} = \cos\beta\underline{i} + \sin\beta\underline{j}$

We know that



$\vec{OB} \times \vec{OA} = |\vec{OB}|\,|\vec{OA}| \sin(\alpha - \beta)\underline{k}$

$\vec{OB} \times \vec{OA} = \sin(\alpha - \beta)\,\underline{k}$....(i)

$$\vec{OB} \times \vec{OA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$\vec{OB} \times \vec{OA} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$|\vec{OB}||\vec{OA}|\sin(\alpha - \beta)\underline{k} = \underline{k}(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$\sin(\alpha - \beta)\,\underline{k} = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\underline{k}$

By (i)

By comparing

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 The non-zero vectors $\underline{a}$ and $\underline{b}$ are parallel of $\underline{a} \times \underline{b} =$ *(Faisalabad Board 2013)*

(a) 1 (b) −1

(c) 0 (d) ab

Q.2 $\underline{U} \times \underline{V}$ is equal to: *(Faisalabad Board 2013)*

(a) UV sinθ (b) $\underline{U} \times \underline{V}$

(c) UV Cos θ (d) $-\underline{V} \times \underline{U}$

Q.3 Volume of parallelepiped with, $\underline{U}$, $\underline{V}$, $\underline{W}$ as its coterminous edges is:

(D.G.K Board 2010, Multan Board 2012 G-I, Faisalabad Board 2013)

(a) $\underline{U} \cdot \underline{V} \times W$ (b) $(\underline{U} \cdot \underline{V} \times \underline{W})$

(c) $\frac{1}{6}(\underline{U} \cdot \underline{V} \times \underline{W})$ (d) $\frac{1}{2}(\underline{U} \cdot \underline{V} \times \underline{W})$

Q.4 The area of triangle whose adjacent sides are $3\underline{i} + 4\underline{j}$ and $12\underline{i} + 9\underline{j}$ is: *(Sargodha Board 2013)*

(a) $\frac{45}{2}$ (b) $\frac{55}{2}$

(c) $\frac{21}{2}$ (d) $\frac{75}{2}$

Q.5 The magnitude of dot and cross product of two vectors are zero and 1 respectively, then angle between vectors is: *(Multan Board 2013)*

(a) 90° (b) 60°

(c) 45° (d) 30°

Q.6 $\underline{j} \times \underline{k}$ = *(Multan Board 2015 G-II)*

(a) $\underline{i}$ (b) $-\underline{i}$

(c) 1 (d) 0

Q.7 If $\underline{U} = 2\underline{i} + 7\underline{j} + 9b$, then $\underline{U} \times \underline{U}$ = *(Multan Board 2012 G-II)*

(a) $19\underline{j}$ (b) $\underline{0}$

(c) $3\underline{i} + 5\underline{j} + 19\underline{k}$ (d) $-4\underline{j}$

Q.8 For two non-zero vectors $\underline{a}$ and $\underline{b}$, $\underline{a} \times \underline{b}$ is: *(Multan Board 2011 G-II)*

(a) ab cosθ (b) ab sinθ

(c) $ab\cos\theta\,\hat{n}$ (d) $ab\sin\theta\,\hat{n}$

Q.9 $\underline{k} \times \underline{i}$ =. *(Gujranwala Board 2014)*

(a) 0 (b) $-\underline{i}$

(c) $-\underline{j}$ (d) $\underline{j}$

Q.10 Vector product of two vectors is a:

(a) Scale quantity (b) Unit vector

(c) Vector quantity (d) Null vector

Q.11 If a and $\underline{b}$ arc non-zero vectors, then $\underline{a} \times \underline{b}$ = *(Lahore Board 2013 G-II)*

(a) $\underline{b} \times \underline{a}$ (b) ab

(c) $\underline{a} - \underline{b}$ (d) $-\underline{b} \times \underline{a}$

Q.12 Commutative law holds in: *(Lahore Board 2013 G-I)*

(a) Vector product

(b) Cross product in three vectors

(c) Inner product

(d) None of these

Q.13 If $\underline{i}$, $\underline{j}$, $\underline{k}$ are unit vectors, then $\underline{k} \times \underline{j}$ = *(Gujranwala Board 2012)*

(a) $\underline{i}$ (b) $-\underline{i}$

(c) 1 (d) 0

Q.14 $(\underline{i} \times \underline{j}) \times \underline{k}$ = *(Bahawalpur Board 2016)*

(a) 0 (b) 1

(c) j (d) $\underline{k}$

Q.15 $2\underline{i} \times \underline{j} - \underline{k}$ = *(Faisalabad Board 2016)*

(a) 2 (b) 0

(c) 1 (d) −2

Q.16 $(\underline{i} \times \underline{k}) \times \underline{j}$ equals *(Lahore Board 2016 G-I)*

(a) −1 (b) 0

(c) 1 (d) ∝

Q.17 $\underline{j} \times \underline{j}$ =

(a) 0 (b) $\underline{j}$

(c) $\underline{i}$ (d) $\underline{k}$

Q.18 $\underline{j} \times \underline{k}$ =

(a) 0 (b) $\underline{j}$

(c) $\underline{i}$ (d) $\underline{k}$

Q.19 If $\underline{u} = [x_1, y_1, z_2]$ and $\underline{v} = [x_2, y_2, z_2]$, then $\underline{u} \times \underline{v}$ = ----------, which is known as determinant formula for $\underline{u} \times \underline{v}$.

(a) $\begin{vmatrix} \underline{i} & \underline{k} & \underline{j} \\ x_1 & z_1 & y_1 \\ x_2 & z_2 & y_2 \end{vmatrix}$ (b) $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$

(c) $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_2 & y_2 & z_2 \\ x_1 & y_1 & z_1 \end{vmatrix}$ (d) None of these

Q.20 Area of parallelogram of two vectors u and v along two adjacent sides of parallelogram is equal to:

(a) $\underline{u} \times \underline{v}$ (b) $|\underline{u} \times \underline{v}|$

(c) $|\underline{u} . \underline{v}|$ (d) $\frac{1}{2}|\underline{u} \times \underline{v}|$

Q.21 Area of triangle = ----------, if $\underline{u}$ and $\underline{v}$ are vectors along two adjacent sides of the triangle. *(Bahawalpur Board 2018)*

(a) $\frac{1}{2}|\underline{u} \times \underline{v}|$ (b) $[\underline{u} \times \underline{v}]$

(c) $\frac{1}{2}[\underline{u} . \underline{v}]$ (d) $\frac{\underline{u} \times \underline{v}}{2}$

Q.22 $\underline{i} \times \underline{j}$ =

(a) 0 (b) $\underline{j}$

(c) $\underline{i}$ (d) $\underline{k}$

Q.23 $\underline{k} \times \underline{j}$ =

(a) 0 (b) $-\underline{i}$

(c) $\underline{i}$ (d) $\underline{k}$

Q.24 $\underline{k} \times \underline{k}$ =

(a) 0 (b) $\underline{j}$

(c) $\underline{i}$ (d) $\underline{k}$

Q.25 $\underline{k} \times \underline{i}$ = *(Bahawalpur Board 2019)*

(a) 0 (b) $\underline{j}$

(c) $\underline{j}$ (d) $\underline{k}$

Q.26 $\underline{i} \times \underline{i}$ is equal to

(a) 0 (b) $\underline{j}$

(c) $\underline{i}$ (d) $\underline{k}$

Q.27 $\underline{j} \times \underline{i}$ =

(a) 0 (b) $\underline{j}$

(c) $-\underline{k}$ (d) $\underline{k}$

Q.28 $\underline{k} \times \underline{i} =$ *(Bahawalpur Board 2019)*

(a) $\underline{k}$ (b) $\underline{j}$

(c) $\underline{i}$ (d) $\underline{k}$

Q.29 If $\underline{a}$ and $\underline{b}$ are parallel vectors, then $\underline{a} \times \underline{b}$ is equal to *(Faisalabad Board 2018)*

(a) 0 (b) 1

(c) $\underline{O}$ (d) None of these

Q.30 If θ is the angle between $\underline{u}$ and $\underline{v}$, then θ is equal to

(a) $\text{Sin}\left\{\frac{|\underline{u} \times \underline{v}|}{|\underline{u}||\underline{v}|}\right\}$ (b) $\text{Sin}^{-1}\left\{\frac{|\underline{u} \times \underline{v}|}{|\underline{u}||\underline{v}|}\right\}$

(c) $\text{Cos}^{-1}\left\{\frac{|\underline{u} \times \underline{v}|}{|\underline{u}||\underline{v}|}\right\}$ (d) $\text{Sin}^{-1}\left\{\frac{|\underline{u} \cdot \underline{v}|}{|\underline{u}||\underline{v}|}\right\}$

Q.31 Which of the following is a vector quantity.

(a) Work (b) Temperature

(c) Distance (d) Displacement

Q.32 $\underline{i} \times \underline{k} =$ *(D.G.K Board 2017 G-I)*

(a) $\underline{i}$ (b) $-\underline{i}$

(c) 0 (d) 1

Q.33 A unit vector ⊥ to $\underline{U}$ and $\underline{V}$ is given.

(Multan Board 2018 G-I)(Bahawalpur Board 2018)

(a) $\underline{U} \times \underline{V}$ (b) $\underline{U} \cdot \underline{V}$

(c) $\frac{\underline{U} \times \underline{V}}{|\underline{U} \times \underline{V}|}$ (d) $|\underline{U} \times \underline{V}|$

Q.34 $\underline{k} \times \underline{i}$ equals:

(Bahawalpur, Faisalabad Board 2019)

(a) $\underline{j}$ (b) $\underline{k}$

(c) 1 (d) 0

Q.35 $|\underline{a} \times \underline{b}|$ calculates the area of:

(Faisalabad Board 2019)

(a) Triangle (b) Parallelogram

(c) Tetrahedron (d) Parallelepiped

Q.36 $\underline{i} \times \underline{k} =$: *(Faisalabad Board 2019 G-II)*

(a) $-\underline{j}$ (b) 1

(c) $\underline{j}$ (d) 0

Q.37 Non-zero vector $\underline{\alpha}$ and $\underline{b}$ are parallel if

$\underline{\alpha} \times \underline{b} =$ *(Lahore Board 2019)*

(a) 0 (b) 1

(c) −1 (d) (a , b)

Q.38 $2\hat{i} \times 2\hat{i} =$ *(Sahiwal Board 2019)*

(a) $4\hat{i}$ (b) 4

(c) $4\hat{k}$ (d) 0

EXERCISE 7.5

SHORT ANSWERS TO THE QUESTIONS

Q.1 Find the volume of the parallelepiped determined by

$\underline{u} = \underline{i} + 2\underline{j} - \underline{k}$, $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$, $\underline{w} = \underline{i} - 7\underline{j} - 4\underline{k}$

(D.G Khan Board 2014 G-II)(Sargodha Board 2018) (Bahawalpur Board 2019)

Ans. Volume of the parallelepiped $= \underline{u} \cdot \underline{v} \times \underline{w}$

$$= \begin{vmatrix} 1 & 2 & -1 \\ 1 & -2 & 3 \\ 1 & -7 & -4 \end{vmatrix}$$

$$= 1(8+21) - 2(-4-3) - 1(-7+2)$$

$$= 1(29) - 2(-7) - 1(-5)$$

$$= 29 + 14 + 5 = 48 \text{ cubic units}$$

Q.2 Find volume of the tetrahedron with the vertices.

$(0,1,2)$, $(3,2,1)$, $(1,2,1)$ **and** $(5,5,6)$

(Lahore Board 2014 G-II)(Rawalpindi Board 2014)

Ans. Let $A(0,1,2)$, $B(3,2,1)$, $C(1,2,1)$

and $D(5,5,6)$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3\underline{i} + 2\underline{j} + \underline{k}) - (0\underline{i} + \underline{j} + 2\underline{k})$$

$$= 3\underline{i} + \underline{j} - \underline{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (\underline{i} + 2\underline{j} + \underline{k}) - (0\underline{i} + \underline{j} + 2\underline{k})$$

$$= \underline{i} + \underline{j} - \underline{k}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (5\underline{i} + 5\underline{j} + 6\underline{k}) - (0\underline{i} + \underline{j} + 2\underline{k})$$

$$= 5\underline{i} + 4\underline{j} + 4\underline{k}$$

$$\therefore \text{Volume of tetrahedron} = \frac{1}{6}\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]$$

$$= \frac{1}{6}\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 5 & 4 & 4 \end{vmatrix}$$

$$= \frac{1}{6}[3(4+4) - 1(4+5) - 1(4-5)]$$

$$= \frac{1}{6}[24 - 9 + 1] = \frac{1}{6}[16] = \frac{8}{3}$$

Q.3 Find the volume of the tetrahedron whose vertices are *(Gujranwala Board 2018)*

$A(2,1,8)$, $B(3,2,9)$, $C(2,1,4)$ and $D(3,3,0)$

Ans. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (3\underline{i} + 2\underline{j} + 9\underline{k}) - (2\underline{i} + \underline{j} + 8\underline{k}) = \underline{i} + \underline{j} + \underline{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (2\underline{i} + \underline{j} + 4\underline{k}) - (2\underline{i} + \underline{j} + 8\underline{k}) = -4\underline{k}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (3\underline{i} + 3\underline{j} + 0\underline{k}) - (2\underline{i} + \underline{j} + 8\underline{k})$

$= \underline{i} + 2\underline{j} - 8\underline{k}$

$\therefore$ Volume of tetrahedron $= \frac{1}{6}\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]$

$= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & -8 \end{vmatrix} = \frac{1}{6}[1(8) - 1(4) + 1(0)] = \frac{1}{6}[8-4]$

$= \frac{1}{6}(4) = \frac{2}{3}$

Q.4 Find the value of α, so that $\alpha\underline{i} + \underline{j}$, $\underline{i} + \underline{j} + 3\underline{k}$ and $2\underline{i} + \underline{j} - 2\underline{k}$ are coplaner.

(Lahore Board 2014 G-I, 2019 G-II) (Sahiwal Board 2018)(Sargodha Board 2019)

Ans. Let $\underline{u} = \alpha\underline{i} + \underline{j}$, $\underline{v} = \underline{i} + \underline{j} + 3\underline{k}$, $\underline{w} = 2\underline{i} + \underline{j} - 2\underline{k}$

Since $\underline{u}$, $\underline{v}$ and $\underline{w}$ are coplaner, so $[\underline{u}\ \underline{v}\ \underline{w}] = 0$

$\Rightarrow \begin{vmatrix} \alpha & 1 & 0 \\ 1 & 1 & 3 \\ 2 & 1 & -2 \end{vmatrix} = 0$

$\Rightarrow \alpha(-2-3) - 1(-2-6) + 0(\ldots) = 0$

$\Rightarrow -5\alpha + 8 = 0 \Rightarrow 5\alpha = 8 \Rightarrow \alpha = \frac{8}{5}$

Q.5 Find α if $\underline{i} - \underline{j} + \underline{k}$, $\underline{i} - 2\underline{j} - 3\underline{k}$ and $3\underline{i} - \alpha\underline{j} + 5\underline{k}$ are coplanar. *(Gujranwala Board 2017) (Faisalabad Board 2017)*

Ans. Let $\underline{u} = \underline{i} - \underline{j} + \underline{k}$, $\underline{v} = \underline{i} - 2\underline{j} - 3\underline{k}$

$\underline{w} = 3\underline{i} - \alpha\underline{j} + 5\underline{k}$

Given vectors are coplanar is

$\underline{u} \cdot \underline{v} \times \underline{w} = 0$

$\begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix} = 0$

$1(-10 - 3\alpha) + 1(5 + 9) + 1(-\alpha + 6) = 0$

$-10 - 3\alpha + 5 + 9 - \alpha + 6 = 0$

$10 - 4\alpha = 0$

$\alpha = \frac{5}{2}$

Q.6 Define "Work".

Ans. Work is the dot (scalar) product of force and displacement.

Q.7 Constant force $\underline{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$ moves an object from $P_1(3,1,-2)$ to $P_2(2,4,6)$. Find the workdone.

(Gujranwala Board 2014)(Rawalpindi Board 2016)

Ans. $\underline{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$; $P_1(3,1,-2)$

and $P_2(2,4,6)$

$\underline{d} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (2\underline{i} + 4\underline{j} + 6\underline{k}) - (3\underline{i} + \underline{j} - 2\underline{k})$

$= -\underline{i} + 3\underline{j} + 8\underline{k}$

Work done $= \underline{F} \cdot \underline{d} = (4\underline{i} + 3\underline{j} + 5\underline{k}) \cdot (-\underline{i} + 3\underline{j} + 8\underline{k})$

$= (4)(-1) + (3)(3) + (5)(8)$

$= -4 + 9 + 40 = 45$ units

Q.8 Define moment of force. (A.J.K Board 2017)

Ans. Moment of force $\vec{F}$ is defined as crors product of $\overrightarrow{OA}$ and $\vec{F}$ where A is point of application of force and O is point about which body is rotated.

Q.9 A force $\mathbf{F} = 7\underline{i} + 4\underline{j} - 3\underline{k}$ is applied at P(1, –2, 3). Find its moment about the point Q(2, 1, 1)

(Sargodha Board 2013)(Bahawalpur Board 2018)

Ans. Here $\mathbf{F} = 7\underline{i} + 4\underline{j} - 3\underline{k}$

$\vec{r} = \overrightarrow{QP}$

$= (1-2)\underline{i} + (-2-1)\underline{j} + (3-1)\underline{k}$

$\vec{r} = -\underline{i} - 3\underline{j} + 2\underline{k}$

Moment $= \vec{r} \times \vec{F}$

$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -3 & 2 \\ 7 & 4 & -3 \end{vmatrix} = \underline{i}(9-8) - \underline{j}(3-14) + \underline{k}(-4+21)$

$= \underline{i} + 11\underline{j} + 17\underline{k}$

Q.10 A force $\vec{F} = 4\underline{i} - 3\underline{k}$, passes through the point $A(2,-2,5)$. Find the moment of $\vec{F}$ about the point $B(1,-3,1)$. *(Bahawalpur Board 2016)*

Ans. $\vec{r} = \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

$= (2\underline{i} - 2\underline{j} + 5\underline{k}) - (\underline{i} - 3\underline{j} + \underline{k}) = \underline{i} + \underline{j} + 4\underline{k}$

M_B = Moment of force about B $= \vec{r} \times \vec{F}$

$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix} = \underline{i}(-3) - \underline{j}(-19) + \underline{k}(-4)$

$= -3\underline{i} + 19\underline{j} - 4\underline{k}$

Q.11 A force $\underline{F} = 2\hat{i} + \hat{j} - 3\hat{k}$ is acting at a point A(1, –2, 1). Find the moment of $\underline{F}$ about point B(2, 0, –2). *(Gujranwala Board 2019 G-II)*

Ans. Let A (1, –2, 1) and B (2, 0, –2)

$$\underline{r} = \overrightarrow{BA} = (1-2)\underline{i} + (-2-0)\underline{j} + (1+2)\underline{k}$$

$$= -\underline{i} - 2\underline{j} + 3\underline{k}$$

$$\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$$

Moment of force about B $= \underline{r} \times \underline{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$$

$$= \underline{i}(6-3) - \underline{j}(3-6) + \underline{k}(-1+4)$$

$$= 3\underline{i} + 3\underline{j} + 3\underline{k}$$

LONG QUESTIONS

Q.1 Find the constant α such that the vectors are coplanar for $\underline{i} - \underline{j} + \underline{k}$, $\underline{i} - 2\underline{j} - 3\underline{k}$ and $2\underline{i} - \alpha\underline{j} + 5\underline{k}$. *(Lahore Board 2015 G-II)*

Sol: See Short Question 12

Q.2 Find the value of α so $\alpha\underline{i} + \underline{j}$, $\underline{i} + \underline{j} + 3\underline{k}$ and $2\underline{i} + \underline{j} - 2\underline{k}$ are co-planar. *(Rawalpindi Board 2016)(Lahore Board 2018 G-II) (Faisalabad Board 2019 G-II)*

Sol: See Short Question 11

Q.3 Find the volume of tetrahedron with vertices (0,1,2), (3,2,1), (1,2,1) and (5,5,6). *(Sahiwal Board 2014, 2015)(Faisalabad Board 2013) (Lahore Board 2013, 2015 G-I, 2014 G-II) (Rawalpindi Board 2016)*

Sol: See Short Question 8

Q.4 Find volume of tetrahedron whose vertices are A(2, 1, 8), B(3, 2, 9), C(2, 1, 4), D(3, 3, 10). *(Lahore Board 2016, 2018 G-I) (Multan Board 2018 G-I, G-II) (Gujranwala Board 2019 G-II)*

Sol: See Short Question 9

Q.5 A force of magnitude 6 units acting parallel to $2\underline{i} - 2\underline{j} + \underline{k}$ displace the point of application from (1, 2, 3) to (5, 3, 7). Find the work done. *(Lahore Board 2014 G-I)*

Sol: Let $\overrightarrow{F_1} = 2\underline{i} - 2\underline{j} + \underline{k}$

$$\left|\overrightarrow{F_1}\right| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$$

let $\hat{F}_1$ be unit vector in the direction of $\overrightarrow{F_1}$

then $\hat{F}_1 = \dfrac{\overrightarrow{F_1}}{\left|\overrightarrow{F_1}\right|} = \dfrac{2\underline{i} - 2\underline{j} + \underline{k}}{3}$

Required force $\underline{F} = 6\hat{F}_1 = 6\left(\dfrac{2\underline{i} - 2\underline{j} + \underline{k}}{3}\right)$

$$= 2(2\underline{i} - 2\underline{j} + \underline{k}) = 4\underline{i} - 4\underline{j} + 2\underline{k}$$

$$\underline{d} = \overrightarrow{AB} = (5-1)\underline{i} + (3-2)\underline{j} + (7-3)\underline{k} = 4\underline{i} + \underline{j} + 4\underline{k}$$

Work done $= \underline{F} \cdot \underline{d}$

$$= (4\underline{i} - 4\underline{j} + 2\underline{k}) \cdot (4\underline{i} + \underline{j} + 4\underline{k})$$

$$= 16 - 4 + 8 = 20 \text{ unit}$$

Q.6 A force $\overrightarrow{F} = 4\underline{i} - 3\underline{k}$ passes through the point A(2, –2, 5). Find the moment of the force about the point B(1, –3, 1). *(Lahore Board 2012 G-I) (Multan Board 2018 G-II)*

Sol: See Short Question 18

Q.7 Find the moment about A(1, 1, 1) of each of the concurrent forces $\underline{i} - 2\underline{j}$, $2\underline{i} + 2\underline{j} - \underline{k}$, $5\underline{j} + 2\underline{k}$ where P(2, 0, 1) is their point of concurrency. *(Multan Board 2016 G-II)(Rawalpindi Board 2019)*

Sol: Let A (1, 1, 1) and P (2, 0, 1)

$$\underline{r} = \overrightarrow{AP} = (2-1)\underline{i} + (0-1)\underline{j} + (1-1)\underline{k} = \underline{i} - \underline{j} + 0\underline{k}$$

$F_1 = \underline{i} - 2\underline{j}$, $F_2 = 3\underline{i} + 2\underline{j} - \underline{k}$, $F_3 = 5\underline{i} + 2\underline{k}$

$$\underline{F} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \underline{i} - 2\underline{j} + 3\underline{i} + 2\underline{j} - \underline{k} + 5\underline{i} + 2\underline{k}$$

$$= 9\underline{i} + 0\underline{j} + \underline{k}$$

Moment of force about B $= \underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 9 & 0 & 1 \end{vmatrix}$

$$= \underline{i}(-1-0) - \underline{j}(1-0) + \underline{k}(0+9) = -\underline{i} - \underline{j} + 9\underline{k}$$

MULTIPLE CHOICE QUESTIONS

❑ ***Each question has four possible answers. Select the correct answer and encircle it.***

Q.1 $2\underline{i} \times 2\underline{j} \cdot \underline{k} =$ *(Sahiwal Board 2013)*

(a) 2 (b) 3

(c) 4 (d) 0

Q.2 $[\underline{k}\ \underline{i}\ \underline{j}] =$ *(Multan Board 2013 G-II, 2018 G-I)*

(a) 0 (b) –1

(c) –2 (d) 1

Q.3 $\underline{j} \cdot \underline{k} \times \underline{i}$ is equal to *(Rawalpindi Board 2014)*
(a) $\underline{i}$ (b) 1
(c) −1 (d) $\underline{j}$

Q.4 $[\underline{i}\ \underline{j}\ \underline{k}]$ is equal to: *(Rawalpindi Board 2014)*
(a) 1 (b) 0
(c) −1 (d) 2

Q.5 $2\underline{i} \times 2\underline{j} \cdot \underline{k} =$ *(Bahawalpur Board 2014)*
(a) −4 (b) 4
(c) 1 (d) 0

Q.6 $\underline{i} \cdot (\underline{j} \times \underline{k})$ is equal to:
(Multan Board 2014 G-I, D.G.K Board 2015 G-I)
(Rawalpindi Board 2017 G-II)
(a) 0 (b) 1
(c) −1 (d) 2

Q.7 The vector $\vec{a}$ lies in the plane of vector $\vec{b}$ and $\vec{c}$, then $\vec{a} \cdot \vec{b} \times \vec{c} =$ *(D.G.K Board 2014, G-II)*
(a) 0 (b) 1
(c) −1 (d) 2

Q.8 $\underline{j} \cdot \underline{k} \times \underline{i}$ equals: *(Sahiwal Board 2014)*
(a) $\underline{i}$ (b) $\underline{j}$
(c) 1 (d) $\underline{k}$

Q.9 The moment of a force $\vec{F}$ acting at P about C is:
(D.G.K Board 2015 G-II)
(a) $\vec{F} \times \vec{CP}$ (b) $\vec{CP} \times \vec{F}$
(c) $\vec{CP} \cdot \vec{F}$ (d) $\vec{OP} \times F$

Q.10 The value of $[\underline{k}\ \underline{i}\ \underline{j}]$ equals to:
(D.G.K Board 2010)
(a) 1 (b) −1
(c) 0 (d) 2

Q.11 If $\underline{a} = [1, 0, 0]$, $\underline{b} = [0, 2, 0]$ and $\underline{c} = [0, 0, 3]$, then $\underline{a} . \underline{b} \times \underline{c}$ is: *(Multan Board 2011 G-I)*
(a) 1 (b) 2
(c) 3 (d) 6

Q.12 $2\underline{i}$, $(2\underline{j} \times \underline{k})$ equals. *(Lahore Board 2014 G-I, GII)*
(a) 0 (b) 2
(c) 4 (d) 6

Q.13 Work done by a constant force $\underline{F}$ during displacement $\underline{d}$ is equal to.
(Lahore Board 2014 G-I)
(a) $\vec{F} \cdot \vec{d}$ (b) $\underline{F} \times \underline{d}$
(c) $\underline{r} \times \underline{F}$ (d) $\underline{F} \times \underline{r}$

Q.14 Moment of force $\underline{F}$ about $(\underline{r})$ is.
(Lahore Board 2015 G-II)
(a) $\underline{r} \times \underline{F}$ (b) $\vec{\underline{F}} \times \underline{r}$
(c) $\underline{r} . \underline{f}$ (d) $\underline{F} . \underline{i}$

Q.15 $\underline{j} . (\underline{k} \times \underline{i})$ is equal to:
(a) 0 (b) −1
(c) 1 (d) 2

Q.16 $\underline{u} \times (\underline{v} . \underline{w})$ is: *(Lahore Board 2013 G-I)*
(a) Scalar product (b) Vector product
(c) Cross product (d) Meaningless

Q.17 If any two vectors of scalar triple product are equal, then its value is:
(Gujranwala Board 2012)(Faisalabad Board 2018)
(a) 1 (b) −1
(c) 2 (d) 0

Q.18 The value of $\underline{k} . \underline{i} \times \underline{j}$ or $[\underline{k}\ \underline{i}\ \underline{j}]$:
(Lahore Board 2012 G-II) (D.G.K Board 2017 G-I)
(a) 0 (b) 1
(c) 2 (d) −1

Q.19 $[\underline{i}\ \underline{j}\ \underline{k}] =$ *(Lahore Board 2012 G-II)*
(a) 1 (b) 3
(c) 0 (d) 2

Q.20 $\underline{j} . \underline{k} \times \underline{i} = 0$ *(Sargodha Board 2016)*
(Faisalabad Board 2017)
(a) $\underline{i}$ (b) $\underline{j}$
(c) $\underline{k}$ (d) 1

Q.21 What is the value of $[\underline{a}\ \underline{b}\ \underline{b}] =$
(Rawalpindi Board 2016)
(a) 1 (b) −1
(c) 0 (d) 2

Q.22 The ---------- is $\frac{1}{6}$ of the volume of the parallelepiped.
(a) Volume of the tetrahedron
(b) Volume of the parallelepiped
(c) Volume of the triangle
(d) None of these

Q.23 If any two vectors of scalar triple product are equal, then its value is equal to:
(a) 0 (b) 2
(c) 1 (d) None of these

Q.24 $\frac{1}{6}$ $[\underline{u}\ \underline{v}\ \underline{w}]$ is the
(a) Volume of the tetrahedron
(b) Volume of the parallelepiped
(c) Volume of the triangle
(d) None of these

Q.25 The vectors $\underline{u}$, $\underline{v}$ and $\underline{w}$ are coplanar if and only if $[\underline{u}\ \underline{v}\ \underline{w}]$ is equal to:

(a) 0 (b) 2

(c) 1 (d) None of these

Q.26 The scalar triple product $(\underline{u} \times \underline{v}) . \underline{w}$ represents the ---------- having $\underline{u}$, $\underline{v}$ and $\underline{w}$ as its conterminous edges.

(a) Volume of the tetrahedron

(b) Volume of the parallelepiped

(c) Volume of the sphere

(d) None of these

Q.27 Vector triple product of three non-zero vector $\vec{a}$, $\vec{b}$ and $\vec{c}$ is denoted by.

(Rawalpindi Board 2017 G-I)

(a) $\vec{a} \times (\vec{b} \times \vec{c})$ (b) $\vec{a} . \vec{b} \times \vec{c}$

(c) $\underline{a} . (\underline{b} + \underline{c})$ (d) $\underline{a} . (\underline{b} - \underline{c})$

Q.28 $\frac{1}{6}$ $[\underline{u}\ \underline{v}\ \underline{w}]$ is formula to calculate.

(Sargodha Board 2017)

(a) Area of triangle (b) Area of parallelogram

(c) Volume of parallelepiped

(d) Volume of tetrahedral

Q.29 $2\underline{i} . \underline{j} \times 2\underline{k} =$ *(Lahore Board 2017 G-I)*

(a) 0 (b) 2

(c) 1 (d) 4

Q.30 $[\underline{a}\ \underline{b}\ \underline{a}]$ is equal to. *(Lahore Board 2017 G-II)*

(a) 1 (b) a

(c) 0 (d) b

Q.31 If $\vec{U} = \vec{V}$, then $\vec{V} . (\vec{V} \times \vec{W})$.

(Gujranwala Board 2018)

(a) 0 (b) 1

(c) −1 (d) Can not be calculated

Q.32 $2\underline{i}(3\underline{j} \times \underline{k})$ is equal to. *(Rawalpindi Board 2018)*

(a) 0 (b) 2

(c) 4 (d) 6

Q.33 $[\underline{i}\ \underline{j}\ \underline{k}] =$ *(Sahiwal Board 2018)*

(a) 1 (b) 2

(c) 0 (d) −1

Q.34 The triple scalar product of vectors, calculates the volume of: *(Lahore Board 2019)*

(a) Triangle (b) Parallelogram

(c) Tetrahedron (d) Parallelepiped

Q.35 Moment of force $\underline{F}$ about a point with position vector $\underline{r}$ will be equal to:

(Sargodha Board 2019)

(a) $\underline{F} \times \underline{r}$ (b) $\underline{r} \times \underline{F}$

(c) $\underline{d} \times \underline{F}$ (d) $\underline{F} \times \underline{d}$

BAHWALPUR BOARD 2019

MATHEMATICS INTERMEDIATE PART-II

(Objective Type)

Total Marks: 20 **Time Allowed: 30 minutes**

Q.1 Four possible answers A, B, C and D to each question are given. The choice which you think is correct, fill that circle in front of that question with Marker or Pen ink on the answer book. Cutting or filling two or more circles will result in zero mark in the question.

(1) Projection of $\vec{u} = a\,\underline{i} + b\,\underline{j} + c\,\underline{k}$ along $\underline{i}$ is:

(a) b (b) a (c) c (d) a + b

(2) $\underline{K} \times \underline{i}$ equals:

(a) $\underline{j}$ (b) $\underline{k}$ (c) 1 (d) 0

(3) Slope of tangent to parabola $y^2 = 4ax$ at (a, 2a) is:

(a) 3 (b) 2 (c) -1 (d) 1

(4) Focus of the Parabola $x^2 = 4ay$ is:

(a) (a , 0) (b) (-a , 0) (c) (0 , a) (d) (0 , -a)

(5) The length of diameter of the circle $x^2 + y^2 - 4x - 12 = 0$ is:

(a) 6 (b) 7 (c) 8 (d) 9

(6) The Graph of the inequality $ax + by < c$ is:

(a) Circle (b) Parabola (c) Straight Line (d) Half Plane

(7) The perpendicular distance of the line $3x + 4y + 5 = 0$ from the origin is:

(a) 0 (b) 1 (c) 2 (d) 5

(8) Equation of the line having slope - 5 and y - intercept - 7 is:

(a) $5x + y + 7 = 0$ (b) $5x - y + 7 = 0$ (c) $5x + y - 7 = 0$ (d) $7x + y + 5 = 0$

(9) When a line intersects the y - axis at (0, 4) then y - intercept is:

(a) 4 (b) 2 (c) 0 (d) 6

(10) Slope of the Line Perpendicular to line $2x - 3y + 1 = 0$ is equal to:

(a) $\frac{3}{2}$ (b) $-\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

(11) Solution of Differential Equation $\frac{dy}{dx} = Sec^2x$ is:

(a) $y = Cot\,x + c$ (b) $y = \tan x + c$ (c) $y = Cos\,x + c$ (d) $y = -\tan x + c$

(12) If $\int_2^k 2\,dx = 12$, then K = ?

(a) 12 (b) 16 (c) 8 (d) 4

(13) $\int \frac{dx}{\sqrt{5 - x^2}} =$:

(a) $\text{Sin}^{-1}\frac{5}{x}$ (b) $\text{Sin}^{-1}\frac{x}{\sqrt{5}}$ (c) $\text{Sin}^{-1}\frac{x}{5}$ (d) $\text{Sin}^{-1}\frac{\sqrt{5}}{x}$

(14) $\int \sec^2 x \tan x \, dx =$

(a) $\text{Sec } x \tan^2 x + c$ (b) $\frac{\text{Sec}^3 x}{3} + C$ (c) $\frac{\text{Sec}^3 x \tan x}{3}$ (d) $\frac{\tan^2 x}{2} + C$

(15) If $y = 1n^{ex}$, then $\frac{dy}{dx} =$:

(a) e^x (b) $\frac{1}{e^x}$ (c) 1 (d) e^{x-1}

(16) The derivative of x^3 w.r.t x^2 is equal to:

(a) $\frac{3x^2}{2}$ (b) $\frac{3x}{2}$ (c) $\frac{2}{3x}$ (d) $\frac{2}{3x^2}$

(17) $\frac{d}{dx}(2x^2 + 3)^5 =$

(a) $(2x^2 + 3)^4 \, 20x$ (b) $20(2x^2 + 3)^5$ (c) $15(2x^2 + 3)^5$ (d) $(2x^2 + 3)^5 \, 100x$

(18) Which one is leibniz Notation for Derivative of f(x) :

(a) $\frac{df}{dx}$ (b) f(x) (c) $\frac{d}{dx}$ (d) Df(x)

(19) $\lim_{x \to -1} \frac{x^3 - x}{x + 1}$

(a) 0 (b) ∞ (c) 2 (d) 1

(20) If P is perimeter of square and A is area then P = :

(a) $2\sqrt{A}$ (b) 4A (c) $4\sqrt{A}$ (d) A^2

BAHWALPUR BOARD 2019

MATHEMATICS INTERMEDIATE PART-II

Examination Session 2013-2017

(*Subjective Type*)

Total Marks: 80 **Time Allowed: 2:30 minutes**

SECTION-I

Q.2 Write answers of any EIGHT questions: **(16)**

(i) Express the area "A" of a circle as a function of its circumference "C".

(ii) Define odd function and give an example.

(iii) Prove that $\lim_{x\to 0} \frac{\sqrt{x+a}-\sqrt{a}}{x} = \frac{1}{2\sqrt{a}}$.

(iv) Find the derivative of $f(x) = c$ by definition.

(v) If $y = x^4 + 2x^2 + 2$ prove that $\frac{dy}{dx} = 4x\sqrt{y-1}$.

(vi) Find $\frac{dy}{dx}$ if $y = \sqrt{x+\sqrt{x}}$.

(vii) Differentiate $\cos^{-1}\frac{x}{a}$ w.r.t 'x'.

(viii) Differentiate $x^2 \sec 4x$ w.r.t 'x'.

(ix) Find $\frac{dy}{dx}$ if $y = a^{\sqrt{x}}$.

(x) Find y_2 if $y = 2x^5 - 3x^4 + 4x^3 + x - 2$.

(xi) Find $\frac{dy}{dx}$ if $y = x\, e^{\sin x}$.

(xii) Find $\frac{dy}{dx}$ if $y = \frac{x}{\ln x}$.

Q.3 Write answers of any EIGHT questions: **(16)**

(i) Find δy and dy if $y = \sqrt{x}$ when x changes from 4 to 4.41.

(ii) Find the area above the x-axis and under the curve $y = 5 - x^2$ from $x = -1$ to $x = 2$.

(iii) Graph the solution set of linear inequality $2x + 1 \geq 0$ in xy – plane.

(iv) Using differentials find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ if $xy + x = 4$.

(v) Define the definite integral.

(vi) Solve the differential equation $ydx + xdy = 0$.

(vii) Evaluate $\int \frac{\cos x}{\sin x \ln \sin x}\, dx$.

(viii) Evaluate $\int x \ln x\, dx$.

(ix) Define corner point of solution region.

(x) Evaluate $\int \sec^4 x\, dx$.

(xi) Evaluate $\int_{-2}^{0} \frac{1}{(2x-1)^2}$.

(xii) Solve $\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}$.

Q.4 Write answers of any NINE questions: **(18)**

(i) Show that the points A(0, 2), B($\sqrt{3}$, 1) and C(0, –2) are vertices of a right triangle.

(ii) Find equation of the line through (–4, 7) and parallel to $2x - 7y + 4 = 0$.

(iii) Find angle from line with slope $\frac{-7}{3}$ to line with slope $\frac{5}{2}$.

(iv) Find length of tangent from the point P(–5, 10) to circle $5x^2 + 5y^2 + 14x + 12y - 10 = 0$.

(v) Find vertex of parabola $(x - 1)^2 = 8(y + 2)$.

(vi) Find equation of hyperbola with Foci (±4, 0), Vertices (±2, 0).

(vii) If $\overrightarrow{AB} = \overrightarrow{CD}$, find A if B(1, 2), C(–2, 5), D(4, 11) are given points.

(viii) If $\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k}$, $\underline{V} = \underline{i} + \alpha\underline{j} + 3\underline{k}$ are perpendicular vectors, find value of α.

(ix) Find vector perpendicular to each of vectors $\underline{a} = 2\underline{i} + \underline{j} + \underline{k}$, $\underline{b} = 4\underline{i} + 2\underline{j} - \underline{k}$.

(x) Find volume of parallelepiped determined by.
$\underline{u} + \underline{i} + 2\underline{j} - \underline{k}$, $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$, $\underline{w} = \underline{i} - 7\underline{j} - 4\underline{k}$.

(xi) Define trapezium.

(xii) Define ellipse.

(xiii) Define directional angles.

SECTION-II

Note: **Attempt any THREE questions:**

Q.5 **(a)** Discuss the continuity of f(x) at x = 2 **(5)**

$$f(x) = \begin{cases} x^2 - 1 & x < 2 \\ 3 & x \geq 2 \end{cases}$$

(b) Discuss the function $f(x) = \sin x + \frac{1}{2\sqrt{2}} \cos 2x$ for extreme values in the interval $(0, 2\pi)$ **(5)**

Q.6 **(a)** Evaluate the integral $\int \frac{2x^2}{(x-1)^2(x+1)}\, dx$. **(5)**

(b) Find the point three-fifth of the way along the line segment from A(–5, 8) to B(5, 3). **(5)**

Q.7 **(a)** Solve the differential equation $\frac{dy}{dx} + \frac{2xy}{2y+1} = x$. **(5)**

(b) Minimize $z = 3x + y$ subject to the constraints. **(5)**
$3x + 5y \geq 15$
$x + 6y \geq 9$
$x \geq 0$, $y \geq 0$

Q.8 **(a)** Find a joint equation of the lines through the origin and perpendicular to the lines **(5)**
$x^2 - 2xy \tan a - y^2 = 0$.

(b) Find equation of the circle of radius 2 and tangent to the line **(5)**
$x - y - 4 = 0$ at A(1, –3)

Q.9 **(a)** Find the center, Foci, eccentricity, vertices and equations of diretrices of. **(5)**
$\frac{x^2}{4} - \frac{y^2}{9} = 1$

(b) Find a unit vector perpendicular to the plane containing vectors **(5)**
$\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}$ and $\underline{b} = 4\underline{i} + 3\underline{j} - \underline{k}$
Also find sine of the angle between them.

FAISALABAD BOARD 2019 (G-I)

MATHEMATICS INTERMEDIATE PART-II

(*Objective Type*)

Total Marks: 20 **Time Allowed: 30 minutes**

Q.1 Four possible answers A, B, C and D to each question are given. The choice which you think is correct, fill that circle in front of that question with Marker or Pen ink in the answer book. Cutting or filling two or more circles will result in zero mark in the question.

(1) The distance between two points A(-8 , 3), B(2 , -1) is:

(a) 116 (b) (-6 , 2) (c) $2\sqrt{29}$ (d) $\sqrt{58}$

(2) The line 2x - y - 4 = 0 cuts x-axis at point:

(a) (2 , 0) (b) (0 , -2) (c) (0, -4) (d) (4 , 0)

(3) The equation of line $\frac{x}{a} + \frac{y}{b} = 1$ is:

(a) Normal form (b) Intercepts form (c) Point-slope form (d) Two-points form

(4) The perpendicular distance of a line 12x + 5y - 7 = 0 from origin is:

(a) $\frac{1}{13}$ (b) $\frac{13}{7}$ (c) $\frac{7}{13}$ (d) 13

(5) The solution of the inequality 2x + y < 5 is:

(a) (1 , 2) (b) (2 , 1) (c) (2 , 3) (d) (5 , 9)

(6) The center of the circle $(x - 1)^2 + (y + 3)^2 = 9$ is:

(a) (-1 , 3) (b) (-1 , -3) (c) (1 , 3) (d) (1 , - 3)

(7) The vertex of the parabola $y^2 = 16x$ is:

(a) (0 , 0) (b) (4 , 0) (c) (0 , 1) (d) (1 , 1)

(8) The end-points of minor axis of an ellipse are called:

(a) Foci (b) Vertices (c) Covertices (d) Centre

(9) $\hat{k} \times \hat{i} =$:

(a) $2\hat{i}$ (b) $-\hat{i}$ (c) $\hat{j}$ (d) $-\hat{j}$

(10) $|\underline{a} \times \underline{b}|$ calculates the area of:

(a) Triangle (b) Parallelogram (c) Tetrahedron (d) Parallelepiped

(11) The function $x^2 + xy + y^2 = 2$ is a / an:

(a) Constant function (b) Even function (c) Implicit function (d) Explicit function

(12) If $f(x) = 2x - 8$, then $f^{-1}(x) =$:

(a) 8 - 2x (b) 8 + 2x (c) $\frac{x + 8}{2}$ (d) $\frac{x - 8}{2}$

(13) $\frac{d}{dx}(f(0)) =$:

(a) f ' (u) (b) f (du) (c) f ' (u) $\frac{du}{dx}$ (d) f ' (u) du

(14) If $y = \cosh x$, then $\frac{dy}{dx} =$:

(a) - sinh y (b) sinh y (c) - cosh x (d) sinh x

(15) $\frac{d}{dx}(\lambda n \cos x) =$:

(a) tan x (b) cot x (c) - tan x (d) - cot x

(16) $\frac{1}{\sqrt{x^2 - 1}}$ is derivative of:

(a) $\sinh^{-1} x$ (b) $\cosh^{-1} x$ (c) $\tanh^{-1} x$ (d) $\text{corth}^{-1} x$

(17) $\int_0^x 3t^2\, dt =$:

(a) t^3 (b) $\frac{t^3}{3}$ (c) x^3 (d) 0

(18) $\int \lambda nx\, dx =$:

(a) $\frac{1}{x}$ (b) $\frac{(\lambda nx)^2}{2}$ (c) $x\lambda nx$ (d) $x\lambda nx???????$

(19) $\int (4 - x^2)^{-\frac{1}{2}} (-2x)\, dx =$:

(a) $2\sqrt{4 - x^2}$ (b) $\frac{1}{2}\sqrt{4 - x^2}$ (c) $\lambda n (4 - x^2)$ (d) $\lambda n \sqrt{4 - x^2}$

(20) $\int e^x (\cos x + \sin x)\, dx =$:

(a) $e^x \cos x$ (b) $e^x \sin x$ (c) $e^x \tan x$ (d) $\lambda n(\sin x)$

FAISALABAD BOARD 2019 (G-I)

MATHEMATICS INTERMEDIATE PART-II

Examination Session 2013-2017

(*Subjective Type*)

Total Marks: 80 **Time Allowed: 2:30 minutes**

SECTION-I

Q.2 Write answers of any EIGHT questions: **(16)**

(i) Define exponential function.

(ii) $f(x) = 2x + 1$, $g(x) = x^2 - 1$, find $g(f(x))$.

(iii) Prove the identity $\cosh^2 x + \sinh^2 x = \cosh 2x$.

(iv) Find by definition derivative of $\frac{1}{x-a}$.

(v) Differentiate $\frac{(x^2+1)^2}{x^2-1}$ w.r.t. x.

(vi) Find $\frac{dy}{dx}$ by making suitable substitution if $y = \sqrt{x + \sqrt{x}}$.

(vii) Prove that $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$.

(viii) Differentiate $\sin^2 x$ w.r.t. $\cos^4 x$.

(ix) Find $\frac{dy}{dx}$ if $y = e^{2x} \sin 2x$.

(x) Find y_2 if $y = \sqrt{x} + \frac{1}{\sqrt{x}}$.

(xi) Apply the Maclaurin series, prove $e^{2x} = 1 + 2x + 2x^2 + \ldots$

(xii) Determine the interval in which f is increasing or decreasing if $f(x) = 4 - x^2$, $x \in (-2, 2)$.

Q.3 Write answers of any EIGHT questions: **(16)**

(i) Find δy and dy of function $f(x) = x^2$ when $x = 2$ and $dx = 0.01$.

(ii) Using differential find $\frac{dy}{dx}$ if $xy - \lambda nx = c$

(iii) Evaluate $\int (x+1)(x-3)\,dx$.

(iv) Evaluate $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$.

(v) Evaluate $\int \frac{1}{1+\cos x}\,dx, \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$.

(vi) Evaluate $\int \frac{x^2}{4+x^2}\,dx$.

(vii) Evaluate $\int x \lambda nx\,dx$.

(viii) Evaluate $\int x \sin x\,dx$.

(ix) Evaluate $\int_{-1}^{3} (x^3 + 3x^2)\,dx$.

(x) Evaluate $\int_{0}^{3} \frac{dx}{x^2+9}$.

(xi) Define objective function.

(xii) Graph the solution set of linear inequality $2x + y \leq 6$.

Q.4 Write answers of any NINE questions: **(18)**

(i) Find the point trisecting the join of A(−1, 4) and B(6, 2).

(ii) Find an equation of the line through A(−6, 5) having slope 7.

(iii) Find the point of intersection of the lines $x - 2y + 1 = 0$ and $2x - y + 2 = 0$.

(iv) Define the homogenous equation.

(v) Find the radius of the circle $x^2 + y^2 - 6x + 4y + 13 = 0$.

(vi) Find the equation of axis and focus of parabola $x^2 = -16y$.

(vii) Find the foci of the ellipse $25x^2 + 9y^2 = 225$.

(viii) Find the equations of directrices of hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

(ix) Find the vector from point A to the origin where $\overrightarrow{AB} = 4\underline{i} - 2\underline{j}$ and B is the point (−2, 5).

(x) Define the direction cosines of a vector.

(xi) Find a unit vector in the direction of $\vec{V} = \underline{i} + 2\underline{j} - \underline{k}$.

(xii) Find a scalar 'α' so that the vectors $2\underline{i} + a\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\underline{k}$ are perpendicular.

(xiii) If $\vec{a} + \vec{b} + \vec{c} = 0$ then prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

SECTION-II

Note: **Attempt any THREE questions:**

Q.5 **(a)** Find m and n so that the given function f is continuous at $x = 3$ **(5)**

$$f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x + 0 & \text{if } x > 3 \end{cases}$$

(b) If $y = (\cos^{-1} x)^2$, prove that $(1 - x_2)\, y_2 - xy_1 - 2 = 0$. **(5)**

Q.6 **(a)** Evaluate $\int \frac{x - 2}{(x + 1)(x^2 + 1)}\, dx$. **(5)**

(b) The average entry test score of engineering candidates was 592 in the year 1998 while the score was 564 in 2002. Assuming that the relationship between time and score is linear, find the average score for 2006. **(5)**

Q.7 **(a)** Find the area bounded by curve $y = x^3 - 4x$ and the x-axis. **(5)**

(b) Maximize: $f(x, y) = 2x + 5y$ subject to

Constraints: $2y - x \leq 8$, $x - y \leq 4$, $x \geq 0$, $y \geq 0$. **(5)**

Q.8 **(a)** The vertices of a triangle are A(−2, 3), B(−4, 1) and C(3, 5). Find coordinates of the orthocenter of the triangle. **(5)**

(b) Show that the lines $3x - 2y = 0$ and $2x + 3y - 13 = 0$ are tangents to the circle $x^2 + y^2 + 6x - 4y = 0$ **(5)**

Q.9 **(a)** Find the equations of tangent and normal to the conic $\frac{x^2}{8} + \frac{y^2}{9} = 1$ at the point $\left(\frac{8}{3}, 1\right)$. **(5)**

(b) Prove that $\cos(\alpha + \beta) = \cos\alpha \cos\beta \ \sin\alpha \sin\beta$. **(5)**

AISALABAD BOARD 2019 (G-II)

MATHEMATICS INTERMEDIATE PART-II

(*Objective Type*)

Total Marks: 20 **Time Allowed: 30 minutes**

Q.1 Four possible answers A, B, C and D to each question are given. The choice which you think is correct, fill that circle in front of that question with Marker or Pen ink on the answer book. Cutting or filling two or more circles will result in zero mark in the question.

(1) The position vector of any point in xy-plane is:

(a) $x\underline{i} + y\underline{j} + z\underline{k}$ (b) $y\underline{j} + z\underline{k}$ (c) $x\underline{i} + y\underline{j}$ (d) $x\underline{i} + z\underline{k}$

(2) $\underline{i} \times \underline{k} =$ **:**

(a) $-\underline{j}$ (b) 1 (c) $\underline{j}$ (d) 0

(3) The two separate parts of hyperbola are called:

(a) Foci (b) Vertices (c) Directrices (d) Branches

(4) The latus rectum of the parabola $y^2 = -4ax$ is:

(a) $x = a$ (b) $x = -a$ (c) $y = a$ (d) $y = -a$

(5) Two circles are said to be concentric circles if they have:

(a) Same radius (b) Different center (c) Same center (d) Same diameter

(6) (1 , 0) is not the solution of the inequality:

(a) $x - 3y < 0$ (b) $7x + 2y < 8$ (c) $3x + 5y < 7$ (d) $4x - 3y < 9$

(7) Intercept form of equation of a line is:

(a) $\frac{x}{a} - \frac{y}{b} = 0$ (b) $\frac{x}{a} + \frac{y}{b} = 0$ (c) $\frac{x}{a} - \frac{y}{b} = 4$ (d) $\frac{x}{a} + \frac{y}{b} = 1$

(8) Distance of the point (-2 , 3) from y-axis is:

(a) 2 (b) - 2 (c) 3 (d) - 3

(9) Slope of the line parallel to x-axis is:

(a) Undefined (b) 1 (c) 0 (d) -1

(10) The lines through origin represented by $ax^2 + 2hxy + by^2 = 0$ are coincident if:

(a) $h^2 = ab$ (b) $h^2 + ab = 0$ (c) $h^2 - ab > 0$ (d) $h^2 - ab < 0$

(11) $\int \tan x \sec^2 x \, dx =$ **:**

(a) $\tan x + c$ (b) $\sec^2 x + c$ (c) $\sec x + c$ (d) $\frac{\tan^2 x}{2} + c$

(12) $\int_0^1 \frac{1}{1+x^2}\,dx =$:

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

(13) $\int e^x \left[\sinh^{-1} x + \frac{1}{\sqrt{1+x^2}}\right] dx =$:

(a) $e^x \cosh^{-1} x$ (b) $e^x \cos^{-1} x$ (c) $e^x \sinh^{-1} x$ (d) $e^x \sin^{-1} x$

(14) $\int x^{-1}\,dx =$:

(a) $\lambda nx + c$ (b) $\frac{x^{-2}}{2}$ (c) $-x^2$ (d) 0

(15) $\frac{d}{dx}(\tan^{-1} 3x) =$:

(a) $\frac{1}{1+3x}$ (b) $\frac{3}{1+3x}$ (c) $\frac{1}{1+9x^2}$ (d) $\frac{3}{1+9x^2}$

(16) $\frac{d}{dx}(e^{\sin x}) =$:

(a) cos x (b) $e^{\sin x}\cos x$ (c) $e^{\sin x}\sin x$ (d) sin x

(17) If $y = \lambda nx$ then $y_2 =$:

(a) $\frac{1}{x}$ (b) $\frac{-1}{x}$ (c) $\frac{-1}{x^2}$ (d) $\frac{1}{x^2}$

(18) The notation used for derivative of f(x) by Cauchy is:

(a) Df (x) (b) f'(x) (c) f(x) (d) $\frac{df}{dx}$

(19) $\lim_{n \to +\infty} \left(1+\frac{1}{n}\right)^{2n} =$:

(a) e (b) e^2 (c) e^4 (d) e^6

(20) Cosh 2x = :

(a) $\frac{e^{2x} - e^{-2x}}{2}$ (b) $\frac{e^{2x} + e^{-2x}}{2}$ (c) $\frac{e^x - e^{-2x}}{2}$ (d) $\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$

FAISALABAD BOARD 2019 (G-II)

MATHEMATICS INTERMEDIATE PART-II

Examination Session 2013-2017

(*Subjective Type*)

Total Marks: 80 **Time Allowed: 2:30 minutes**

SECTION-I

Q.2 Write answers of any EIGHT questions: **(16)**

(i) Define implicit function.

(ii) Prove the identity $\operatorname{sech}^2 x = 1 - \tanh^2 x$.

(iii) Find $\lim_{n \to 0} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$, $x > 0$.

(iv) If $y = x^4 + 2x^2 + 2$, prove that $\frac{dy}{dx} = 4x\sqrt{y-1}$.

(v) Differentiate w.r.t. x if $y = \frac{2x-3}{2x+1}$.

(vi) Differentiate $x^2 - \frac{1}{x^2}$ w.r.t. x^4.

(vii) Prove that $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

(viii) Find $\frac{dy}{dx}$ if $y = x \cos y$.

(ix) Differentiate $y = x^{\sqrt{x}}$.

(x) Find $\frac{dy}{dx}$ if $y = \lambda n (\tanh x)$.

(xi) Define point of inflextion of a function.

(xii) Determine $f(x) = \sin x$ is increasing or decreasing in the interval $\left(0, \frac{\pi}{2}\right)$.

Q.3 Write answers of any EIGHT questions: **(16)**

(i) Find δy and $dy = y = \sqrt{x}$, when x changes from 4 to 4.41.

(ii) Evaluate $\int \sin^2 x \, dx$.

(iii) Integrate by substitution $\int \frac{x}{\sqrt{4+x^2}} dx$.

(iv) Find the integral $\int \frac{\sqrt{2}}{\sin x + \cos x} dx$.

(v) Evaluate the integral by parts $\int \lambda nx \, dx$.

(vi) Find indefinite integral $\int \frac{1}{\sqrt{a^2 - x^2}} dx$ by substitution.

(vii) Evaluate $\int \frac{2a}{x^2 - a^2} dx$, $x > a$ by partial fraction.

(viii) What is the definition of definite integral?

(ix) Calculate the integral $\int_{-1}^{5} |x - 3| \, dx$.

(x) Define order of a differential equation.

(xi) What do you know about half planes?

(xii) Graph the linear inequality $2x + 3 \geq 0$.

Q.4 Write answers of any NINE questions: (18)

(i) Find the point P on the join of A(1, 4) and B(5, 6) that is twice as far from A as B is from A and lies on the same side of A as B does.

(ii) Show that the points A(–3, 6), B(3, 2) and C(6, 0) are collinear.

(iii) Find an equation of the line through the points A(–5, –3) and B(9, –1).

(iv) Find separate equations of line represented by $6x^2 - 19xy + 15y^2 = 0$.

(v) Define eccentricity of the conic.

(vi) Find equation of parabola with focus (–1, 0), vertex (–1, 2).

(vii) Find equation of hyperbola with foci (±5, 0), vertex (3, 0).

(viii) Define a circle.

(ix) Find sum of vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ if A(1, –1), B(2, 0), C(–1, 3), D(–2, 2).

(x) Find a vector whose magnitude is 2 and is parallel to $-\underline{i} + \underline{j} + \underline{k}$.

(xi) Find a scalar 'α' so that the vectors $2\underline{i} + \alpha\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\underline{k}$ are perpendicular.

(xii) Find area of triangle formed by P, Q, R if P(0, 0, 0), Q(2, 3, 2), R(–1, 1, 4).

(xiii) Find α so that $\alpha\underline{i} + \underline{j}$, $\underline{i} + \underline{j} + 3\underline{k}$ and $2\underline{i} + \underline{j} + 2\underline{k}$ are coplanar.

SECTION-II

Note: **Attempt any THREE questions:**

Q.5 **(a)** Prove that $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$; $a > 0$. (5)

(b) If $x = a(\theta - \sin\theta)$; $y = a(1 + \cos\theta)$ then prove that $y2\frac{d^2y}{dx^2} + a = 0$. (5)

Q.6 **(a)** Evaluate $\int \tan^3 x \sec x \, dx$. (5)

(b) Find the equations of two parallel lines perpendicular to $2x - y + 3 = 0$ such that the product of the x and y intercepts of each is 3. (5)

Q.7 **(a)** Evaluate $\int_0^{\sqrt{3}} \frac{x^4 + 9x + 1}{x^2 + 9} dx$. (5)

(b) Indicate the solution region of the following system of linear inequalities by shading.

$3x + 7y \leq 21$, $2x - y \geq -3$, $x \geq 0$ (5)

Q.8 **(a)** Find an equation of the line through the intersection of

$16x - 10y - 33 = 0$, $12x + 14y + 29 = 0$ and the intersection of

$x - y - 4 = 0$, $x - 7y + 2 = 0$ (5)

(b) Write the equations of tangent and normal to the circle $x^2 + y^2 = 25$ at the point (4, 3). (5)

Q.9 **(a)** Show that the ordinate at any point P of the parabola is mean proportional between the length of Latus rectum and abscissa of P. (5)

(b) Prove that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. (5)

GUJRANWALA BOARD 2019 (G-I)

MATHEMATICS INTERMEDIATE PART-II

(*Objective Type*)

Total Marks: 20 **Time Allowed: 30 minutes**

Q.1 Four possible answers A, B, C and D to each question are given. The choice which you think is correct, fill that circle in front of that question with Marker or Pen ink in the answer book. Cutting or filling two or more circles will result in zero mark in the question.

(1) If A(-3, 6) and B(3, 2), then slope of AB is

(a) $\frac{3}{2}$ (b) $\frac{-2}{3}$ (c) $\frac{1}{3}$ (d) $\frac{-3}{2}$

(2) $\int_a^b 3t^2\,dt =$

(a) $a^3 - b^3$ (b) $a^3 + b^3$ (c) $b^3 - a^3$ (d) $\frac{b^3 + a^3}{3}$

(3) If $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, then $\overrightarrow{AB} =$

(a) $\vec{a} - \vec{b}$ (b) $\vec{a} + \vec{b}$ (c) $\vec{b} - \vec{a}$ (d) $\vec{b} + \vec{a}$

(4) Minimum value of the function $f(x) = x^2 + 2x - 3$ is at x =

(a) - 3 (b) 1 (c) 0 (d) -1

(5) The range of $f(x) = x^2$ is

(a) $(-\infty, 0)$ (b) $(-\infty, \infty)$ (c) $(-1, 0)$ (d) $(0, \infty)$

(6) $|\cos\alpha\,\underline{i} + \sin a\,\underline{j} + 0\underline{k}| =$

(a) 0 (b) – 1 (c) 2 (d) 1

(7) The length of tangent from (0 , 1) to the circle $x^2 + y^2 + 6x - 3y + 3 = 0$ is

(a) 2 (b) 3 (c) 4 (d) 1

(8) (1, -3) is in the solution of region

(a) $x + y > 0$ (b) $x + y < 0$ (c) $x + y = 0$ (d) $x - y = 0$

(9) $\frac{d}{dx}(\sinh 2x) =$

(a) 2 cosh 2x (b) 2 sinh 2x (c) -2 cosh 2x (d) - 2 sinh 2x

(10) Centre of the circle $5x^2 + 5y^2 + 14x + 12y - 10 = 0$ is

(a) $\left(\frac{-7}{5}, \frac{-6}{5}\right)$ (b) $\left(\frac{7}{5}, \frac{6}{5}\right)$ (c) (7 , 6) (d) (7, -6)

(11) **If $f(x) = \cos x$, then $f^2\left(\frac{\pi}{2}\right)$**

(a) - 1 (b) $-\frac{1}{2}$ (c) 0 (d) 1

(12) **Anti derivative of cot x =**

(a) λn cos x + c (b) λb sub x + c (c) - λn cos x + c (d) - λn sin x + c

(13) $\frac{d}{dx}(\cos^{-1} 3x) =$

(a) $\frac{3}{\sqrt{1-9x^2}}$ (b) $\frac{-3}{\sqrt{1-9x^2}}$ (c) $\frac{1}{\sqrt{1-9x^2}}$ (d) $\frac{-1}{\sqrt{1-9x^2}}$

(14) **Focus of parabola $x^2 = -16y$ is**

(a) (0, -4) (b) (0, 4) (c) (4, 0) (d) (-4, 0)

(15) $\int_{-1}^{0} \frac{1}{1+x^2}\, dx =$

(a) $\frac{\pi}{4}$ (b) $\frac{4}{\pi}$ (c) $-\frac{\pi}{4}$ (d) $-\frac{4}{\pi}$

(16) **Centroid of triangle with vertices A(2, 1), B(-1, 3) and C(-1, -4) is**

(a) (3, 1) (b) (0, 0) (c) (2, 2) (d) (-2, -5)

(17) $\int e^{\tan x} \sec^2 x\, dx =$

(a) $-e^{\tan x} + c$ (b) $e^{\tan x} + c$ (c) $e^{\tan 2x} + c$ (d) $e^{\cos x} + c$

(18) **Distance between (1, 2) and (2, 1) is**

(a) $\sqrt{3}$ (b) $\sqrt{5}$ (c) $\sqrt{2}$ (d) 7

(19) **Equation of a straight line passing through P(-2, 3) and parallel to x-axis is**

(a) x = - 2 (b) y = 3 (c) x = 3 (d) y = -2

(20) $\frac{d}{dx}\left(\frac{1}{x^2}\right)$ **at x = 1 is**

(a) - 2 (b) 2 (c) 1 (d) -1

GUJRANWALA BOARD 2019 (G-I)

MATHEMATICS INTERMEDIATE PART-II

Examination Session 2013-2017

(*Subjective Type*)

Total Marks: 80 **Time Allowed: 2:30 minutes**

SECTION-I

Q.2 Write answers of any EIGHT questions: **(16)**

(i) Determine whether $f(x) = x\sqrt{x^2+5}$ is even or odd.

(ii) For the real valued function $f(x) = \frac{2x+1}{x-1}$ find $f^{-1}(x)$ and $f^{-1}(-1)$.

(iii) If $f(x) = \begin{cases} x-1, & x<3 \\ 2x+1, & 3 \le x \end{cases}$ find $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$.

(iv) Find the derivative of $f(x) = c$ by first principle.

(v) Differentiate $y = \frac{a+x}{a-x}$ w.r.t, x.

(vi) Find $\frac{dy}{dx}$ if $y = e^{x^2+1}$.

(vii) Determine the values of x, for which $f(x) = x^2 + 2x - 3$ is extreme.

(viii) Show that $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$.

(ix) If $y = \sin^{-1}\frac{x}{a}$ then $\frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}$.

(x) Define a stationary point.

(xi) Define even function and give an example.

(xii) Find $\frac{dy}{dx}$ if $y = \tan h\,(x^2)$.

Q.3 Write answers of any EIGHT questions: **(16)**

(i) Use differentials, find $\frac{dy}{dx}$ if $x^2 + 2y^2 = 4$.

(ii) Evaluate $\int \cos 3x . \sin 2x\, dx$.

(iii) Evaluate $\int \frac{\sin\theta}{1+\cos^2\theta}\, d\theta$.

(iv) Integrate $\tan^{-1} x$ w.r.t, 'x'.

(v) Evaluate $\int e^x(\cos x + \sin x)\, dx$.

(vi) Evaluate $\int_{-1}^{2} (x + |x|)\, dx$.

(vii) Find area between x-axis and curve $y = 4x - x^2$.

(viii) Solve differential equation $xdy + y(x-1)dx = 0$.

(ix) Define order of differential equation.

(x) Evaluate $\int \frac{(1-\sqrt{x})^2}{\sqrt{x}}\, dx$.

(xi) Define corner point.

(xii) Graph the feasible region of $3x - 2y \ge 6$.

Q.4 Write answers of any NINE questions: **(18)**

(i) Show that points A(3, 1), B(−2, −3) and C(2, 2) are vertices of an isosceles triangle.

(ii) Define centroid of a triangle.

(iii) Find an equation of line through A(−6, 5) and having slope 7.

(iv) Convert into two intercept form $2x - 4y + 11 = 0$.

(v) Find center and radius of circle $5x^2 + 5y^2 + 14x + 12y - 10 = 0$.

(vi) Determine whether the point P(−5, 6) lies outside, on or inside the circle $x^2 + y^2 + 4x - 6y - 12 = 0$.

(vii) Write an equation of parabola with focus (−1, 0), vertex (−1, 2).

(viii) Find an equation of ellipse with center (0, 0), focus (0, −3) and vertex (0, 4).

(ix) Define direction angles.

(x) If O is origin and $\overline{OP} = \overline{AB}$, find the point P where A and B are (−3, 7) and (1, 0) respectively.

(xi) Find a vector whose magnitude is 4 and is parallel to $2\underline{i} - 3\underline{j} + 6\underline{k}$.

(xii) Find a and b so that the vectors $3\underline{i} - \underline{j} + 4\underline{k}$ and $a\underline{i} + b\underline{j} - 2\underline{k}$ are parallel.

(xiii) Find a scalar α so that the vector $2\underline{i} + \alpha\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\underline{k}$ are perpendicular.

SECTION-II

Note: **Attempt any THREE questions:**

Q.5 **(a)** Prove that $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$. (5)

(b) Apply the Maclaurin series expansion to prove $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ (5)

Q.6 **(a)** Evaluate the integral $\int \frac{(a-b)x}{(x-a)(x-b)}\,dx$. (5)

(b) Find an equation of the perpendicular bisector of the segment joining the points A(3, 5) and B(9, 8). (5)

Q.7 **(a)** Find the integral $\int_0^{\sqrt{7}} \frac{3x}{\sqrt{x^2+9}}\,dx$. (5)

(b) Graph the feasible region of the inequalities and find the corner points. (5)

$x + y \leq 5$

$-2x + y \geq 2$

$x \geq 0, y \geq 0$

Q.8 **(a)** Show that the lines $4x - 3y - 8 = 0$; $3x - 4y - 6 = 0$; $x - y - 2 = 0$ are concurrent and third line bisect the angle formed by first two. (5)

(b) Find equation of circle which passes through the points A(5, 10), B(6, 9) and C(−2, 3). (5)

Q.9 **(a)** Find the equation of "Ellipse" with vertices (−1, 1); (5, 1) and Foci (4, 1) and (0, 1). (5)

(b) Using vectors, find the area of triangle ABC whose vertices are A(1, −1, 1); B(2, 1, −1) and C(−1, 1, 2). (5)

GUJRANWALA BOARD 2019 (G-II)

MATHEMATICS INTERMEDIATE PART-II

(*Objective Type*)

Total Marks: 20 **Time Allowed: 30 minutes**

Q.1 Four possible answers A, B, C and D to each question are given. The choice which you think is correct, fill that circle in front of that question with Marker or Pen ink in the answer book. Cutting or filling two or more circles will result in zero mark in the question.

(1) If $f(x) = \frac{1}{x^2}$ $(x \neq 0)$, then of fof(x) is

(a) x^4 (b) x^2 (c) 1 (d) $\frac{1}{x^4}$

(2) $\lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^{2n} =$

(a) e (b) e^2 (c) e^n (d) zero

(3) If $y = \frac{1}{x^2}$ then $\frac{dy}{dx}$ at x = -1

(a) 3 (b) $\frac{1}{3}$ (c) 2 (d) $\frac{1}{2}$

(4) $\frac{d}{dx}(\lambda n\ 2x) =$

(a) $\frac{1}{2x}$ (b) $\frac{1}{x}$ (c) $-\frac{1}{2x}$ (d) 2x

(5) If f '(c) = 0 then f(x) has relative maximum value at x = c if

(a) f " (c) < 0 (b) f "(c) > 0 (c) f " (c) = 0 (d) f " (c) = 0

(6) $y = \sin 3x$ then y_2 is

(a) 9 cos x (b) -9 sin 3x (c) 9 sin 3x (d) - 9 cos 3x

(7) $\int \cot x\ dx =$

(a) $\text{cosec}^2 x + c$ (b) $-\text{cosec}^2 x + c$ (c) $\lambda n \sin x + c$ (d) $\lambda n \cos x + c$

(8) $\int_0^{\pi} \sec x \tan x\ dx =$

(a) 0 (b) 1 (c) -1 (d) -2

(9) Order of the differential equation $\frac{x^2 dy}{dx^2} + \frac{dy}{dx} + 2x = 0$ is

(a) 0 (b) 1 (c) 2 (d) 3

(10) $\int \tan^2 x\,dx$ **is equal to:**

(a) $\tan x + x + c$ (b) $\tan x - x + c$ (c) $2\tan x + c$ (d) $2\tan x + x + c$

(11) **Slope of the line** $2x + y - 3 = 0$ **is**

(a) 2 (b) $\frac{2}{3}$ (c) -2 (d) $-\frac{2}{3}$

(12) **The vertices of a triangle are (a, b - c), (b, c - a)(e, a – b) then its centroid is**

(a) $\left(0 + \frac{a+b+c}{3}\right)$ (b) $\left(0 + \frac{a-b-c}{3}\right)$ (c) (0 , 0) (d) $\left(\frac{a+b+c}{3}, 0\right)$

(13) **The perpendicular distance of the line** $12x + 5y = 7$ **from the origin is**

(a) $\frac{7}{13}$ (b) $\frac{13}{7}$ (c) 13 (d) $\frac{1}{13}$

(14) **The point of concurrency of altitudes of a triangle is called**

(a) centroid (b) orthocentre (c) in centre (d) circum centre

(15) **(1, 0) is the solution of the inequality**

(a) $7x + 2y < 8$ (b) $x - 3y < 0$ (c) $10x + 5y < 6$ (d) $-3x + 5y > 2$

(16) **The centre of the circle** $x^2 + y^2 - 6x + 4y + 13 = 0$ **is**

(a) (3, 2) (b) (3, -2) (c) (2, 3) (d) (-2, -3)

(17) **Axis of parabola** $x^2 = 4ay$ **is**

(a) $x = 0$ (b) $y = 0$ (c) $y = x$ (d) $x = -y$

(18) **Eccentricity of an ellipse is**

(a) $e = 1$ (b) $e > 1$ (c) $0 < e < 1$ (d) $e = 0$

(19) **Angle between the vectors** $4\underline{i} + 2\underline{j} - \underline{k}$ **and** $-\underline{i} + \underline{j} - 2\underline{k}$ **is**

(a) 30° (b) 45° (c) 90° (d) 60°

(20) **If** α, β, γ **be the direction angles of a vector then** $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$

(a) 2 (b) 0 (c) -1 (d) 1

GUJRANWALA BOARD 2019 (G-II)

MATHEMATICS INTERMEDIATE PART-II

Examination Session 2013-2017

(*Subjective Type*)

Total Marks: 80 **Time Allowed: 2:30 minutes**

SECTION-I

Q.2 Write answers of any EIGHT questions: **(16)**

(i) Define implicit function.

(ii) If $f(x) = 2x + 1$ and $g(x) = \frac{3}{x-1}$, $x \neq 1$, find fog (x).

(iii) Evaluate $\lim_{x \to -1} \frac{x^3 - x}{x + 1}$ by using algebraic technique.

(iv) Find $\frac{dy}{dx}$ if $y = (x - 5)(3 - x)$.

(v) Find $\frac{dy}{dx}$ if $xy + y^2 = 2$.

(vi) Differentiate $\sin x$ w.r.t. $\cot x$.

(vii) Find $\frac{dy}{dx}$ if $y = \frac{x}{\lambda n x}$.

(viii) Define the stationary point.

(ix) Find $\frac{dy}{dx}$ if $y = e^{-2x} \sin 2x$.

(x) Differentiate $\cot^{-1} \frac{x}{a}$ w.r.t x.

(xi) Find y_2 if $y = \sqrt{x} + \frac{1}{\sqrt{x}}$.

(xii) Find the extreme values for $f(x) = 5x^2 - 6x + 2$.

Q.3 Write answers of any EIGHT questions: **(16)**

(i) Using differentials find $\frac{dx}{dy}$ if $x^2 + 2y^2 = 16$.

(ii) Define first order differential equation.

(iii) Evaluate $\int \tan^2 x \, dx$.

(iv) Evaluate $\int \frac{\sqrt{2}}{\sin x + \cos x} dx$.

(v) Evaluate $\int \sin^{-1} x \, dx$.

(vi) Evaluate $\int \frac{ex(1+x)}{(2+x)^2} dx$.

(vii) Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos t \, dt$.

(viii) Find the area between x-axis and curve $y = \sin 2x$ from $x = 0$ to $x = \frac{\pi}{3}$.

(ix) Solve the differential equation $\frac{x^2 + 1}{y + 1} = \frac{x}{y} \frac{dy}{dx}$ $(x, y > 0)$.

(x) Evaluate $\int x^2 \lambda n x \, dx$.

(xi) Define problem constraints.

(xii) Graph the solution set of linear inequality $3y - 4 \leq 0$ in xy-plane.

Q.4 Write answers of any NINE questions: **(18)**

(i) Find the point that divides the join of A(−6, 3) and B(5, −2) in the ratio 2 : 3 internally.

(ii) A point P(5, 3) is in xy-coordinates system. Axes are rotated through angle 45°.

(iii) Find an equation of line passing through (2, 3) having slope −1.

(iv) Find the point of intersection of the lines $x + 4y - 12 = 0$ and $x - 3y + 3 = 0$.

(v) Find the center and radius of the circle $4x^2 + 4y^2 - 8x + 12y - 25 = 0$.

(vi) Determine the length of tangent drawn from point (−5, 4) to the circle $5x^2 + 5y^2 - 10x + 15y - 131 = 0$

(vii) Find the focus and directrix of the parabola $x^2 = 4(y - 1)$.

(viii) Find the center and eccentricity of the ellipse $\frac{(2x-1)^2}{16} + \frac{(y+2)^2}{16} = 1$.

(ix) Define scalar product of two vectors.

(x) Find a vector of length 5 in the direction opposite to that of $\underline{v} = \hat{i} - 2\hat{j} + 3\hat{k}$.

(xi) Find a vector perpendicular to the plane containing vectors $\underline{a} = 2\hat{i} - 2\hat{j} - 3\hat{k}$, $\underline{b} = 4\hat{i} + 3\hat{j} - \hat{k}$.

(xii) A force $\underline{F} = 2\hat{i} + \hat{j} - 3\hat{k}$ is acting at a point A(1, −2, 1). Find the moment of $\underline{F}$ about point B(2, 0, −2).

(xiii) What are direction angles of a vector?

SECTION-II

Note: **Attempt any THREE questions:**

Q.5 **(a)** If $f(x) = \begin{cases} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2}, & x \neq 2 \\ K, & x = 2 \end{cases}$ find value of K so that f(x) is continuous at x = 2. (5)

(b) Find $\frac{dy}{dx}$ if $y = \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}}$, $x \neq 0$. (5)

Q.6 **(a)** Evaluate $\int \frac{1+4x}{(x-3)(x^2+4)} dx$. (5)

(b) If (4, −2), (−2, 4) and (5, 5) are vertices of a triangle, find the co-ordinates of its "incenter". (5)

Q.7 **(a)** Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos\theta + \sin\theta}{2\cos^2\theta} d\theta$. (5)

(b) Graph the solution region and find the corner points of (5)
$3x + 2y \geq 6$; $x + y \leq 4$; $x \geq 0$, $y \geq 0$.

Q.8 **(a)** Show that the line $2x + 3y - 13 = 0$ is tangent to the circle $x^2 + y^2 + 6x - 4y = 0$. (5)

(b) Prove that the angle in a semi-circle is a right angle. (5)

Q.9 **(a)** Show that an equation of parabola with focus at (a cos α, a sin α). (5)

(b) Find the volume of the tetrahydron whose vertices are
A(2, 1, 8), B(3, 2, 9), C(2, 1, 4), D(3, 3, 10). (5)

LAHORE BOARD 2019

MATHEMATICS INTERMEDIATE PART-II

(*Objective Type*)

Total Marks: 20 **Time Allowed: 30 minutes**

Q.1 Four possible answers A, B, C and D to each question are given. The choice which you think is correct, fill that circle in front of that question with Marker or Pen ink in the answer book. Cutting or filling two or more circles will result in zero mark in the question.

(1) The solution of the inequality $x + 2y < 6$ is:

(a) (1 , 1) (b) (1 , 3) (c) (1 , 4) (d) (1 , 5)

(2) The mid point of line segment joining A(–8, 3), B(2, – 1) is:

(a) (–6 , 2) (b) (10 , 4) (c) (1 , 4) (d) (1 ,5)

(3) Non-zero vector $\underline{a}$ and $\underline{b}$ are parallel if $\underline{a} \times \underline{b} =$:

(a) 0 (b) 1 (c) -1 (d) (a , b)

(4) The perpendicular distance of a line $5x + 12y = 7$ from origin is:

(a) $\frac{1}{13}$ (b) $\frac{13}{7}$ (c) $\frac{7}{12}$ (d) - 7

(5) The triple scalar product of vectors, calculates the volume of:

(a) Triangle (b) Parallelogram (c) Tetrahedron (d) Parallelepiped

(6) y-intercept of the line $2x - y - 4 = 0$ is:

(a) 2 (b) -2 (c) 4 (d) - 4

(7) An angle in the semi circle is of measure:

(a) 30° (b) 60° (c) 90° (d) 180°

(8) The radius of circle $x^2 + y^2 = 5$ is:

(a) 25 (b) $\sqrt{5}$ (c) 5 (d) (0, 0)

(9) The equation of line $\frac{x}{b} + \frac{y}{a} = 1$ is in:

(a) Normal form (b) Intercept form (c) Point-slope form (d) Two-points form

(10) Equation of latus-rectum of parabola $y^2 = 4ax$ is:

(a) x = - a (b) y = - a (c) x = a (d) y = a

(11) A function $f(x)$ has relative maximum at x = c, if $f'(c) = 0$ and:

(a) $f''(c) > 0$ (b) $f''(c) < 0$ (c) $f''(c) = 0$ (d) $f'(c) \neq 0$

(12) The function $y = 27 + x^2$ is a / an:

(a) Constant function (b) Even function (c) Implicit function (d) Explicit function

(13) $\int_{-\pi}^{\pi} \sin x \, dx =$:

(a) 2π (b) 0 (c) 1 (d) $\cos\pi$

(14) $\frac{d}{dx}(\sqrt{x}) =$:

(a) $\sqrt{x}$ (b) $\frac{1}{\sqrt{x}}$ (c) $\frac{1}{2x}$ (d) $\frac{1}{2\sqrt{x}}$

(15) $\int \frac{e^x}{e^x + 3} \, dx =$:

(a) $\lambda n(e^x + 3) + c$ (b) $e^{2x} + c$ (c) $e^0 + c$ (d) $e^{2x} + 3 + c$

(16) $\frac{d}{dx}(\cos x^2) =$:

(a) $2x \sin x^2$ (b) $-2x \sin x^2$ (c) $2 \cos x$ (d) $-2\sin x$

(17) $\int \sec^2 x dx =$:

(a) $\cot x + c$ (b) $\tan x + c$ (c) $2 \sec x + c$ (d) $\frac{1}{\cos^2 x} + c$

(18) **If $f(x) = 2x + 1$, then $f^{-1}(x) =$? :**

(a) $2x - 1$ (b) $1 - 2x$ (c) $x - \frac{1}{2}$ (d) $\frac{x - 1}{2}$

(19) $\int \tan x dx =$:

(a) $\lambda n \left| \sec x \right| + c$ (b) $\lambda n \left| \operatorname{cosec} x \right| + c$ (c) $\lambda n \left| \sin x \right| + c$ (d) $\lambda n \left| \cot x \right| + c$

(20) **If $y = \sin^{-1} \frac{x}{a}$, the $\sin y =$:**

(a) $\cos y$ (b) $\cos x$ (c) $\frac{x}{a}$ (d) $\frac{y}{a}$

LAHORE BOARD 2019

MATHEMATICS INTERMEDIATE PART-II

Examination Session 2013-2017

(*Subjective Type*)

Total Marks: 80 **Time Allowed: 2:30 minutes**

SECTION-I

Q.2 Write answers of any EIGHT questions: **(16)**

(i) Define implicit function.

(ii) $f(x) = 3x^4 - 2x^2$, $g(x) = \frac{2}{\sqrt{x}}$, find $f(g(x))$.

(iii) Evaluate $\lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2}$.

(iv) Find derivative by definition of x^2.

(v) Differentiate w.r.t 'x' $\sqrt{\frac{a - x}{a + x}}$.

(vi) Find $\frac{dy}{dx}$ if $x^2 - 4xy - 5y = 0$.

(vii) Prove that $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1 + x^2}$.

(viii) Find $\frac{dy}{dx}$ if $y = x \cos y$.

(ix) Find $f'(x)$ if $f(x) = \sqrt{\ln(e^{2x} + e^{-2x})}$.

(x) Find y_2 if $x = at^2$, $y = bt^4$.

(xi) Define Maclaurin series.

(xii) Determine the interval in which $f(x)$ is increasing or decreasing if $f(x) = \sin x$, $x \in (0, \pi)$.

Q.3 Write answers of any EIGHT questions: **(16)**

(i) Using differential, find $\frac{dy}{dx}$ when $xy - \ln x = c$.

(ii) Evaluate $\int \frac{(\sin x + \cos^2 x)}{\cos^2 x \cdot \sin x} dx$.

(iii) Find $\int x(\sqrt{x} + 1)dx$; $x > 0$.

(iv) Evaluate $\int a^{x^2} \cdot x \, dx$; $a > 1$.

(v) Find the anti derivative of $x \cdot e^x$.

(vi) Evaluate $\int e^x (\cos x + \sin x) dx$.

(vii) State 'Fundamental Theorem' of calculus.

(viii) Compute $\int_{-1}^{1} (x^{1/3} + 1)dx$.

(ix) Find the area above x-axis and under the curve $y = 5 - x^2$ from $x = -1$ to $x = 2$.

(x) Solve the differential equation $\sin y \cdot \operatorname{cosec} x \cdot \frac{dy}{dx} = 1$.

(xi) Define 'decision variables'.

(xii) Graph solution set of inequality $2x + y \geq 2$ in $x - y$ plane.

Q.4 Write answers of any NINE questions: (18)

(i) Find the coordinates of the point that divides the join of A(−6, 3) and B(5, −2) internally in ratio 2:3.

(ii) Find the slope and inclination of the line joining the point A(−2, 4) and B(5, 11).

(iii) By means of slopes show that point A(−1, −3), B(1, 5) and C(2, 9) are collinear.

(iv) Find equation of the line through (−4, 7) and parallel to the line $2x - 7y + 4 = 0$.

(v) Find equation of circle with center at (5, −2) and radius 4.

(vi) Find focus and vertex of the parabola $y^2 = -8(x - 3)$.

(vii) Find equation of tangent to the parabola $x^2 = 16y$ at the point whose abscissa is 8.

(viii) Find foci and vertices of the ellipse $25x^2 + 9y^2 = 225$.

(ix) Find the angle between the vectors $\underline{u} = 2\underline{i} - \underline{j} + \underline{k}$ and $\underline{v} = -\underline{i} + \underline{j}$.

(x) Find scalar α so that the vectors $2\underline{i} + \alpha\underline{j} + 5\underline{k}$ and $2\underline{i} + \underline{j} + \alpha\underline{k}$ are perpendicular.

(xi) If $\underline{v}$ is a vector for which $\underline{v} \cdot \underline{i} = 0$, $\underline{v} \cdot \underline{j} = 0$, $\underline{v} \cdot \underline{k} = 0$ find $\underline{v}$.

(xii) Prove that $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = 0$.

(xiii) Find the value of α so that $\alpha\underline{i} + \underline{j}$, $\underline{i} + \underline{j} + 3\underline{k}$ and $2\underline{i} + \underline{j} - 2\underline{k}$ are coplanar.

SECTION-II

Note: **Attempt any THREE questions:**

Q.5 **(a)** If $f(x) = \begin{cases} 3x & \text{if } x \le -2 \\ x^2 - 1 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \ge 2 \end{cases}$ discuss continuity at $x = 2$ and $x = -2$. (5)

(b) If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$. (5)

Q.6 **(a)** Integrate $\int \frac{12}{x^3 + 8}\,dx$. (5)

(b) Find equations of two parallel lines, perpendicular to $2x - y + 3 = 0$ such that the product o the x and y-intercepts of each is 3. (5)

Q.7 **(a)** Evaluate the definite integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x}{\sin x(2 + \sin x)}\,dx$. (5)

(b) Minimize $z = 2x + y$ subject to the constraints. (5)

$x + y \ge 3$, $7x + 5y \le 35$, $x \ge 0$, $y \ge 0$.

Q.8 **(a)** Find equation of the line through the point (2, −9) and intersection of the lines (5)

$2x + 5y - 8 = 0$

$3x - 4y - 6 = 0$

(b) Show that the circles $x^2 + y^2 + 2x - 2y - 7 = 0$ and $x^2 + y^2 - 6x + 4y + 9 = 0$ touch externally. (5)

Q.9 **(a)** Find an equation of the ellipse having foci $(\pm 5, 0)$ and passing through the point $\left(\frac{2}{3}, \sqrt{3}\right)$ (5)

(b) A particle acted upon by constant forces $4\underline{i} + \underline{j} - 3\underline{k}$ and $2\underline{i} - \underline{j} - \underline{k}$ is displaced from A(1, 2, 3) to B (5, 4, 1). Find the work done (5)

MULTAN BOARD 2019

MATHEMATICS INTERMEDIATE PART-II

(*Objective Type*)

Total Marks: 20 **Time Allowed: 30 minutes**

Q.1 Four possible answers A, B, C and D to each question are given. The choice which you think is correct, fill that circle in front of that question with Marker or Pen ink on the answer book. Cutting or filling two or more circles will result in zero mark in the question.

(1) **If $f(x) = x^3 + x$, then $f(x)$ is:**

(a) Constant function (b) Even function (c) Odd function (d) Implicit function

(2) $\lim_{x \to 4} \frac{x^2 - 6x + 8}{x - 4} =$

(a) 4 (b) 2 (c) 6 (d) 8

(3) **$x = 3\cos t$, $y = 3\sin t$ represents:**

(a) Line (b) Circle (c) Ellipse (d) Hyperbola

(4) **If $f(x) = \sin x$, then $f''\left(\frac{\pi}{2}\right) =$:**

(a) 0 (b) 1 (c) 2 (d) - 1

(5) $\frac{d}{dx}(\coth x) =$

(a) $-\operatorname{cosech}^2 x$ (b) $\operatorname{cosech}^2 x$ (c) $\tan h^2 x$ (d) $\coth x \operatorname{sec} h x$

(6) $\frac{d}{dx}(e^{x^2})$

(a) e^{x^2} (b) $2e^{x2}$ (c) $2xe^{x2}$ (d) $2e^x$

(7) $\int \frac{\sin 2x}{4 \sin x} dx =$

(a) $\sin 2x + c$ (b) $2\sin 2x + c$ (c) $\frac{1}{2}\sin x + c$ (d) $2\sin x + c$

(8) $\int_1^2 2x\, dx =$

(a) 3 (b) 2 (c) 1 (d) 0

(9) $\int_1^2 \frac{1}{x} dx =$

(a) $2\lambda nx$ (b) $\lambda n2$ (c) λn (d) $\lambda n3$

(10) $\int 5^{2x}\,dx =$

(a) 5^{2x} (b) $2(5^{2x})$ (c) $5^{2x}\lambda n5$ (d) $2(5^{2x}\lambda n5)$

(11) Distance of line $x + 2y + 5 = 0$ from origin is:

(a) 1 (b) $\sqrt{5}$ (c) 5 (d) 2

(12) Length of perpendicular from (1 , 1) to the line $4x - 3y + 9 = 0$ equals:

(a) 2 (b) 4 (c) 3 (d) 9

(13) Equation of horizontal line through (2, 3) is:

(a) $y = 3$ (b) $y = 2$ (c) $x = 3$ (d) $x = 2$

(14) Slope of vertical line is:

(a) 0 (b) 1 (c) ∞ (d) 2

(15) If $3x + 2y \leq 6$, point does not satisfy:

(a) (1 , 0) (b) (0 , 1) (c) (0 , 0) (d) (3 , 2)

(16) Radius of circle $x^2 + y^2 - 4x - 6y = 0$ is:

(a) $\sqrt{13}$ (b) $\sqrt{11}$ (c) $\sqrt{5}$ (d) 13

(17) Directrix of parabola $x^2 = 20y$ is:

(a) $x = 10$ (b) $x = 5$ (c) $y = -5$ (d) $x = -5$

(18) Parabola $x^2 = -8y$ opens:

(a) Rightwards (b) Leftwards (c) Upwards (d) Downwards

(19) Magnitude of vector $6\underline{i} + 3\underline{j} - 2\underline{k}$ is:

(a) 7 (b) 6 (c) 3 (d) -2

(20) Direction cosines of y - axis are:

(a) 0, 0, 1 (b) 1, 0, 0 (c) 0, 1, 0 (d) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Chapter 1

FUNCTION AND LIMITS

EXERCISE 1.1

1	c	2	c	3	b	4	c	5	b
6	d	7	d	8	b	9	a	10	b
11	b	12	b	13	c	14	a	15	b
16	d	17	b	18	d	19	c	20	b
21	a	22	a	23	a	24	c	25	b
26	b	27	a	28	c	29	b	30	d
31	a	32	b	33	c	34	b	35	d
36	c	37	d	38	b	39	b	40	c
41	c	42	a	43	c	44	d	45	a
46	c	47	b	48	c	49	a	50	a
51	d	52	a	53	c	54	c	55	b
56	c	57	a	58	c	59	b	60	d
61	c	62	d	63	c	64	d	65	a
66	c	67	b	68	d	69	a	70	b
71	d	72	c	73	a	74	c	75	a
76	a	77	b	78	c	79	b	80	c
81	d	82	b	83	b	84	b	85	b
86	a	87	b	88	c	89	c	90	a
91	c	92	a	93	c	94	c	95	a
96	c	97	a	98	b	99	c	100	b
101	b	102	d	103	b	104	c	105	b
106	b								

EXERCISE 1.2

1	c	2	b	3	b	4	c	5	c
6	c	7	b	8	b	9	a	10	c
11	d	12	a	13	c	14	a	15	d
16	a	17	b	18	c	19	c	20	b
21	c	22	b	23	c	24	b	25	a
26	a	27	a	28	d	29	a	30	c
31	c	32	c	33	a	34	c	35	d
36	a	37	d	38	a				

EXERCISE 1.3

1	a	2	b	3	b	4	a	5	a
6	d	7	b	8	d	9	a	10	b
11	a	12	d	13	a	14	b	15	d
16	c	17	d	18	b	19	a	20	d
21	d	22	d	23	a	24	d	25	b
26	b	27	a	28	a	29	a	30	b
31	b	32	b	33	b	34	c	35	a
36	a	37	c	38	a	39	c	40	b
41	d	42	c	43	d	44	c	45	a
46	d	47	b	48	d	49	a	50	c
51	c	52	b	53	b	54	b	55	d
56	d	57	c	58	a	59	b	60	a
61	d	62	b	63	b	64	b		

EXERCISE 1.4

1	b	2	c	3	c	4	a	5	a
6	b								

Chapter 2

DIFFERENTIATION

EXERCISE 2.1

1	c	2	b	3	a	4	a	5	c
6	c	7	c	8	c	9	b	10	b
11	b	12	a	13	c	14	d	15	a
16	a	17	c	18	c	19	a	20	d
21	c	22	c	23	a	24	a	25	a

EXERCISE 2.2

1	c	2	b

EXERCISE 2.3

1	c	2	b	3	c	4	b	5	c
6	d	7	d	8	c	9	c	10	b
11	c	12	c	13	a	14	c	15	a
16	a	17	b	18	a	19	c	20	b
21	c	22	c	23	b	24	c	25	b
26	d	27	c	28	a	29	d	30	b
31	b	32	a	33	b	34	b	35	c
36	d	37	b	38	a	39	c	40	a
41	a	42	d	43	d	44	a	45	d
46	a	47	c	48	b	49	d		

EXERCISE 2.4

1	c	2	d	3	a	4	c	5	c
6	b	7	c	8	c	9	a	10	b
11	d	12	b	13	c				

EXERCISE 2.5

1	b	2	a	3	b	4	d	5	b
6	d	7	c	8	b	9	d	10	c
11	d	12	a	13	b	14	c	15	d
16	a	17	c	18	b	19	b	20	a
21	c	22	c	23	b	24	b	25	a
26	d	27	b	28	b	29	c	30	a
31	b	32	c	33	c	34	a	35	b
36	a	37	b	38	c	39	b	40	a
41	d	42	d	43	c	44	b	45	b
46	d	47	b	48	a	49	a	50	b
51	b	52	c	53	d	54	a	55	a
56	b	57	d	58	c	59	b	60	a
61	b	62	b	63	c	64	d	65	c
66	b	67	b	68	b	69	a	70	c
71	b	72	a						

EXERCISE 2.6

1	b	2	d	3	b	4	c	5	b
6	a	7	b	8	c	9	a	10	d
11	d	12	c	13	c	14	b	15	b
16	d	17	c	18	d	19	a	20	c
21	c	22	b	23	a	24	c	25	d
26	d	27	b	28	d	29	a	30	c
31	a	32	c	33	d	34	b	35	a
36	b	37	a	38	c	39	c	40	b
41	a	42	b	43	b	44	a	45	c
46	b	47	a	48	b	49	a	50	b
51	c	52	d	53	b	54	d	55	a
56	a	57	a	58	c	59	a	60	a
61	d	62	a	63	b	64	c	65	b
66	d	67	b	68	c	69	b	70	c
71	d	77	b	73	b	74	c		

EXERCISE 2.7

1	d	2	a	3	a	4	d	5	a
6	d	7	a	8	d	9	c	10	c
11	b	12	c	13	d	14	c	15	d

EXERCISE 2.8

1	a	2	a	3	b	4	a	5	d
6	a	7	d	8	d	9	a	10	a
11	d								

EXERCISE 2.9

1	d	2	b	3	b	4	a	5	d
6	b	7	b	8	d	9	d	10	d
11	b	12	b	13	a	14	a	15	b
16	b	17	a	18	c	19	b	20	a
21	c	22	c	23	c	24	d	25	b
26	d	27	a	28	b	29	b	30	d
31	c	32	a	33	d	34	a	35	b
36	a	37	c	38	a	39	b	40	c
41	a	42	a	43	b	44	c	45	a

Chapter 3

INTEGRATION

EXERCISE 3.1

1	d	2	c	3	b	4	c	5	b

EXERCISE 3.2

1	c	2	d	3	d	4	c	5	a
6	b	7	b	8	d	9	d	10	b
11	c	12	c	13	d	14	a	15	a
16	a	17	c	18	c	19	c	20	c
21	c	22	d	23	c	24	a	25	b
26	a	27	d	28	d	29	c	30	b
31	b	32	d	33	a	34	a	35	b
36	b	37	b	38	b	39	c	40	c
41	d	42	b	43	d	44	a	45	a
46	c	47	d	48	d	49	c	50	d
51	d	52	a	53	b	54	b	55	b
56	c	57	b						

EXERCISE 3.3

1	d	2	c	3	d	4	b	5	d
6	d	7	a	8	b	9	b	10	a
11	a	12	b	13	b	14	c	15	c
16	a	17	a	18	b	19	b	20	b
21	b	22	b	23	d	24	d	25	d
26	c	27	b	28	a	29	c	30	d
31	c	32	c	33	a	34	a	35	c
36	a	37	c	38	c	39	d	40	b
41	c	42	c	43	a	44	b	45	c
46	d	47	a	48	b	49	a	50	a
51	b	52	c	53	d	54	c	55	d
56	c	57	a	58	a	59	d	60	d
61	d	62	c	63	b	64	d	65	a
66	c	67	a	68	b	69	c	70	d
71	b	72	b	73	a	74	b	75	a
76	d	77	a	78	d	79	b	80	a
81	b	82	d	83	b	84	a	85	c
86	b	87	a	88	d	89	b	90	c
91	c	92	a	93	b	94	b	95	a
96	d	97	b	98	a	99	b	100	d
101	c								

EXERCISE 3.4

1	b	2	b	3	a	4	a	5	b
6	a	7	b	8	b	9	d	10	b
11	a	12	a	13	a	14	b	15	a
16	a	17	a	18	d	19	b	20	c
21	c								

EXERCISE 3.5

1	d

EXERCISE 3.6

1	d	2	a	3	a	4	b	5	c
6	d	7	a	8	d	9	c	10	d
11	d	12	b	13	c	14	a	15	d
16	c	17	c	18	d	19	d	20	c
21	b	22	d	23	a	24	b	25	b
26	c	27	d	28	c	29	b	30	b
31	c	32	c	33	a	34	a	35	c
36	b	37	c	38	a	39	b	40	a
41	b	42	c	43	c	44	c	45	c
46	c	47	c	48	b	49	c	50	c
51	c	52	c	53	b	54	c	55	a
56	d	57	b	58	a	59	b	60	a
61	d	62	c						

EXERCISE 3.7

1	b	2	a	3	b	4	b	5	b
6	c								

EXERCISE 3.8

1	b	2	b	3	c	4	a	5	c
6	c	7	a	8	a	9	b	10	a
11	d	12	a	13	c	14	d	15	c
16	a	17	b	18	b	19	a	20	c
21	b	22	c	23	b	24	d	25	b
26	a	27	b	28	b	29	c	30	a
31	b								

Chapter 4

INTRODUCTION TO ANALYTICAL GEOMETRY

EXERCISE 4.1

1	a	2	c	3	c	4	a	5	b
6	a	7	d	8	d	9	a	10	b
11	d	12	b	13	b	14	b	15	d
16	a	17	a	18	b	19	c	20	c
21	b	22	a	23	d	24	d	25	b
26	d	27	b	28	a	29	b	30	c
31	a	32	b	33	c	34	d	35	b
36	b								

EXERCISE 4.2

1	a

EXERCISE 4.3

1	c	2	b	3	b	4	b	5	a
6	a	7	d	8	a	9	c	10	c
11	d	12	c	13	c	14	b	15	a
16	b	17	b	18	b	19	a	20	b
21	c	22	a	23	d	24	a	25	b
26	b	27	c	28	b	29	d	30	a
31	b	32	b	33	d	34	c	35	d
36	c	37	d	38	a	39	c	40	c
41	c	42	c	43	d	44	b	45	b
46	a	47	b	48	a	49	d	50	c
51	c	52	a	53	d	54	a	55	a
56	a	57	b	58	a	59	d	60	b
61	b	62	b	63	a	64	c	65	c
66	a	67	a	68	d	69	d	70	c
71	d	72	a	73	a	74	a	75	a
76	a	77	d	78	a	79	c	80	a
81	a	82	b	83	b	84	a	85	a
86	b	87	b	88	b	89	b	90	d
91	b	92	c	93	d	94	b	95	b
96	c	97	c	98	b	99	a	100	c
101	c	102	b	103	c	104	d	105	b
106	c	107	a	108	d	109	a	110	c
111	a	112	a	113	c	114	d	115	c
116	a	117	d	118	c	119	b	120	a
121	d	122	c	123	a	124	a	125	d
126	a	127	b	128	b	129	a	130	a
131	b	132	c	133	a	134	b	135	c
136	d	137	c	138	b	139	c	140	c
141	d	142	b	143	b	144	a	145	d
146	c	147	d	148	b	149	b	150	d
151	a	152	b	153	d	154	c	155	b
156	d	157	c						

EXERCISE 4.4

1	a	2	d	3	a	4	d	5	a
6	a	7	a	8	b	9	a	10	a
11	c	12	b	13	b	14	b		

EXERCISE 4.5

1	d	2	c	3	b	4	c	5	a
6	a	7	c	8	b	9	c	10	b
11	a	12	d	13	c	14	a	15	c

Chapter 5

LINEAR INEQUALITIES & LINEAR PROGRAMMING

EXERCISE 5.1

1	a	2	a	3	b	4	a	5	b
6	a	7	a	8	a	9	b	10	a
11	b	12	c	13	d	14	d	15	d
16	b	17	c	18	c	19	d	20	a
21	b	22	a	23	a	24	c	25	d
26	a	27	c	28	d	29	c	30	c
31	a	32	b	33	b	34	d	35	b
36	c	37	a	38	c	39	a	40	c
41	a	42	a	43	b	44	b	45	c
46	b	47	c	48	c	49	a	50	a
51	c	52	c	53	a	54	a	55	c
56	d	57	b	58	a	59	d	60	a
61	b	62	c	63	a	64	b	65	c
66	c	67	d	68	b	69	d	70	d
71	a	72	b	73	b	74	a	75	a
76	c	77	d	78	d				

EXERCISE 5.2

1	b	2	c	3	b	4	a	5	b

EXERCISE 5.3

1	d	2	b	3	b	4	b	5	d
6	a								

Chapter 6

CONIC SECTIONS

EXERCISE 6.1

1	d	2	b	3	a	4	c	5	a
6	a	7	d	8	d	9	d	10	c
11	b	12	a	13	d	14	b	15	c
16	d	17	b	18	b	19	d	20	d
21	c	22	b	23	c	24	c	25	b
26	b	27	a	28	c	29	d	30	a
31	b	32	b	33	a	34	b	35	b
36	c	37	b	38	a	39	b	40	c
41	a	42	d	43	a	44	a	45	a
46	c	47	c	48	c	49	a	50	c
51	b	52	d	53	d	54	c	55	a
56	b	57	b	58	a	59	b	60	d
61	a								

EXERCISE 6.2

1	d	2	d	3	d	4	b	5	c
6	a	7	c	8	b	9	c	10	c
11	c	12	c	13	c	14	a	15	d
16	a	17	c	18	c	19	d		

EXERCISE 6.3

1	b	2	b	3	b	4	b

EXERCISE 6.4

1	c	2	b	3	c	4	b	5	c
6	c	7	a	8	d	9	a	10	a
11	a	12	c	13	c	14	d	15	b
16	a	17	b	18	a	19	a	20	c
21	c	22	c	23	a	24	a	25	b
26	b	27	c	28	c	29	d	30	a
31	b	32	a	33	b	34	c	35	b
36	c	37	a	38	c	39	c	40	d
41	d	42	a	43	c	44	c	45	b
46	b	47	d	48	b	49	a	50	c
51	d	52	c	53	b	54	b	55	b
56	c	57	a	58	d	59	b	60	b
61	a	62	a	63	a	64	c	65	b
66	d	67	d	68	d	69	d	70	a
71	c	72	d	73	d	74	c	75	c
76	a	77	b	78	a	79	a	80	c
81	d	82	c	83	a	84	a		

EXERCISE 6.5

1	c	2	d	3	b	4	b	5	c
6	c	7	a	8	c	9	b	10	b
11	d	12	b	13	d	14	a	15	a
16	c	17	c	18	b	19	a	20	a
21	c	22	a	23	a	24	a	25	b
26	a	27	b	28	c	29	b	30	b
31	a	32	b	33	b	34	c	35	c
36	a	37	b	38	a	39	a	40	c
41	a	42	a	43	b	44	a	45	d
46	d	47	c	48	c	49	d	50	c

EXERCISE 6.6

1	d	2	d	3	a	4	b	5	a
6	b	7	a	8	d	9	a	10	c
11	b	12	a	13	c	14	a	15	d
16	c	17	b	18	d	19	c	20	d

EXERCISE 6.7

1	d	2	a	3	b	4	a	5	a
6	a	7	d	8	d	9	c	10	d

EXERCISE 6.8

1	c	2	c	3	b	4	a	5	a
6	c	7	b	8	d	9	c		

Chapter 7

VECTORS

EXERCISE 7.1

1	a	2	d	3	d	4	c	5	c
6	c	7	a	8	a	9	a	10	b
11	d	12	a	13	d	14	b	15	b
16	a	17	a	18	b	19	b	20	c
21	c	22	a	23	b	24	d	25	a
26	b	27	a	28	b	29	b	30	c
31	a	32	c	33	c	34	c	35	c
36	a	37	c	38	d	39	b	40	a
41	b	42	b	43	b	44	d	45	d
46	a	47	d	48	d	49	d	50	d
51	a	52	c	53	a	54	c		

EXERCISE 7.2

1	b	2	a	3	c	4	a	5	c
6	c	7	b	8	b	9	c	10	c
11	a	12	a	13	a	14	b	15	b
16	c	17	a	18	d	19	c	20	d
21	c	22	c	23	a	24	d	25	c
26	b	27	c	28	d	29	d	30	a
31	c								

EXERCISE 7.3

1	b	2	a	3	b	4	b	5	b
6	b	7	b	8	c	9	a	10	b
11	c	12	b	13	c	14	c	15	a
16	a	17	b	18	c	19	a	20	b
21	a	22	b	23	a	24	a	25	b
26	a	27	a	28	a	29	a	30	a
31	a	32	d	33	b	34	c	35	b
36	c	37	d	38	a	39	b	40	a
41	d	42	c	43	c	44	a	45	a
46	c	47	c	48	c	49	d	50	a

EXERCISE 7.4

1	c	2	d	3	a	4	c	5	a
6	a	7	b	8	d	9	d	10	c
11	d	12	c	13	a	14	a	15	a
16	b	17	a	18	c	19	b	20	b
21	a	22	d	23	b	24	a	25	b
26	a	27	c	28	b	29	a	30	b
31	d	32	b	33	c	34	a	35	b
36	a	37	a	38	d				

EXERCISE 7.5

1	c	2	d	3	b	4	a	5	b
6	b	7	a	8	c	9	a	10	a
11	d	12	c	13	a	14	a	15	c
16	d	17	d	18	a	19	a	20	d
21	c	22	a	23	a	24	a	25	a
26	b	27	a	28	d	29	d	30	c
31	a	32	b	33	a	34	d	35	b