Chapter

8

Progression or Series

8.1. Introduction

Examples

When a set of numbers is arranged according to some law in such a manner that there is a first number, a second number, a third number and so on, the set of numbers is called a *progression, series* or *sequence* of numbers. The successive numbers are called the *terms* of the progression.

8.2. Arithmetic Progression

An *Arithmetic Progression* is a sequence of numbers in which each subsequent term starting from the first is obtained by adding a fixed number called the common difference to the preceding number. Thus the following are examples of arithmetic progrestion (A.P.).

1, 3, 5, 7, 9, 11, ... Common difference 2

2, 5, 8, 11, 14, ... Common difference 3

25, 20, 15, 10, 5, 0, −5, ... Common difference −5

In general, an A.P. with 'a' as first term and 'd' as common difference will be

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d,\ \ldots\ldots$$

We have :

1st term $= a+0d = a+(1-1)d$

2nd term $= a+\ d = a+(2-1)d$

3rd term $= a+2d = a+(3-1)d$

4th term $= a+3d = a+(4-1)d$

5th term $= a+4d = a+(5-1)d$

. . . .

. . . .

. . . .

Proceeding in this manner, we will have :

$$n\text{th term} = a+(n-1)d$$

If there are only n terms the nth term will be the last term L given by :

$$L = a+(n-1)d$$

8.3. The sum of an A P.

Let S_n be the sum of first n terms of an A.P. whose first term is 'a' and common difference 'd' then

$$S_n = a+(a+d)+(a+2d)+(a+3d)+\ldots\ldots+(L-d)+L$$

This can also be written in reverse order as

$$S_n = L+(L-d)+(L-2d)+(L-3d)+\ldots\ldots+(a+d)+a$$

Adding these two equations we get

$$2S_n = (a+L)+(a+L)+(a+L)+(a+L)+\ldots\ldots+(a+L)+(a+L)$$

$$= n(a+L) \qquad (a+L \text{ added } n \text{ times})$$

$$S_n = \frac{n}{2}(a+L)\ldots\ldots\ldots\ldots\ldots(2)$$

Now substituting the value of L obtained in (1) above

$$S_n = \frac{n}{2}\{a+a+(n-1)d\}$$

$$S_n = \frac{n}{2}\{2a+(n-1)d\}\ldots\ldots\ldots(3)$$

With the help of the above formulas, if any three of the five variables a, d, n, L and S_n are given the other two can be found.

Example 1 :

(a) Find the 20th term of the series 1, 3, 5, 7, 9 . . . . .

(b) What will be the sum ?

Solution :

(a) We have, $n=20$, $a=1$ and $d=2$.

$$L = a+(n-1)d$$
$$= 1+(20-1)\times 2$$
$$= 39$$

(b) $$S_n = \frac{n}{2}(a+L)$$
$$= \frac{20}{2}(1+39)$$
$$= 400$$

Alternatively,

$$S_n = \frac{n}{2}\{2a+(n-1)d\}$$
$$= \frac{20}{2}\{(2\times 1)+(20-1)\times 2\}$$
$$= 10\,(2+38)$$
$$= 400$$

Example 2 :

The sum of 10 terms of an A.P, whose last term is 28, is 145. Find the first term and the common difference.

Solution :

We have $n=10$, $L=28$ and $S_n=145$

$$S_n=\frac{n}{2}(a+L)$$

$$145=\frac{10}{2}(a+28)$$

$$145=5a+140$$

$$5a=5$$

$$a=1$$

Putting $a=1$, in

$$L=a+(n-1)d$$

$$28=1+(10-1)d$$

$$9d=27$$

$$d=3$$

The first 10 terms of the progression will therefore be 1, 4, 7, 10, 13, 16, 19, 22, 25, 28.

Example 3 :

Find the sum of the series $5\frac{1}{2}$, 4, $2\frac{1}{2}$, 1, $-\frac{1}{2}$, -2 . . . . to 12 terms.

Solution :

We have : $a=5\frac{1}{2}$, $d=4-5\frac{1}{2}=-1\frac{1}{2}$ and $n=12$

$$S_n=\frac{12}{2}\{2\times5\tfrac{1}{2}+(12-1)(1-1\tfrac{1}{2})\}$$

$$=6\{2\times\frac{11}{2}-11\times\frac{3}{2}\}$$

$$=6\{11-\frac{33}{2}\}$$

$$=6\times(\frac{-11}{2})=-33$$

8.4. Arithmetic Mean

The terms between any two given terms of an A.P. are called the *arithmetic means.*

If a, A and b are in A.P., then

$$A-a=b-A$$

$$2A=a+b$$

$$A=\frac{a+b}{2}$$

Here A is the arithmetic mean between 'a' and 'b'.

In order to find out a given number of arithmetic means between two terms, the farmula $L=a+(n-1)\ d$ can be used, where n is the number of terms to be inserted plus 2.

Example 4 :

(a) What is arithmetic mean between 5 and 17 ?

(b) Insert five arithmetic means between 5 and 17 ?

Solution :

(a) $A.M. = \frac{5+17}{2} = 11$

(b) Here $a=5$, $L=17$ and $n=5+2=7$

Applying the formula

$$L = a+(n-1)d$$
$$17 = 5+(7-1)d$$
$$12 = 6d$$
$$d = 2$$

Thus the required means are :

$$5+2=7$$
$$7+2=9$$
$$9+2=11$$
$$11+2=13$$
$$13+2=15$$

The A.P will be 5, 7, 9, 11, 13, 15 and 17 having the required five means between 5 and 17.

Example 5 :

Pipe pieces each 25 centimetres in diameter and 12 meters in length are required for a gas pipe line. In the supply yard the pipes are stacked in layers with each layer containing one less pipe piece than the layer beneath it. If one stack contains 10 layers with 28 pipe pieces in the top layer, does the yard contain enough pipe pieces to complete a three and a half kilometre long pipe line ?

Solution :

To find the total number of pipe pieces, we have to find the sum of an A.P. with $a=28$, $n=10$ and $d=1$.

Total number of pipe pieces is given by

$$S = \frac{n}{2}\{2a+(n-1)d\}$$
$$= \frac{10}{2}\{2\times 28+(10-1)\times 1\}$$
$$= 5(56+9)$$
$$= 325$$

Total length of 325 pipe pieces $= 325 \times 12$ metres $= 3900$ metres

$= 3.90$ kilometres

Therefore, the stock of pipes in the yard is sufficient to lay more than 3.50 kilometre pipe line.

8.5. Geometric Progression

A *geometric progression* is a sequence of numbers arranged in a definite order such that each number after the first can be found by multiplying the preceding number by a constant factor called the common ratio. The abbreviation used for geometric progression is G P. Thus 1, 2, 4, 8, 16 . . . . . is a G.P. with a common ratio of 2.

If 'a' be the first term, 'r' the common ratio and 'n' the number of terms of a G.P., then the terms of the sequence will be

$$a,\ ar,\ ar^2,\ ar^3 \ldots\ldots ar^{n-1}$$

$$\frac{ar}{a}=\frac{ar^2}{ar}=\frac{ar^3}{ar^2}=\frac{ar^{n-1}}{ar^{n-2}}=\frac{ar^{n-1}}{ar^{n-2}}=r \text{ (common ratio)}$$

The nth term of the series is given by

$$L=ar^{n-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Let S_n be the sum of the G P. upto n terms. Then

$$S_n=a+ar+ar^2+ar^3+\ldots\ldots+ar^{n-1} \ldots\ldots\ldots (i)$$

Multiplying both sides of the equation by 'r', we get

$$r\,S_n=ar+ar^2+ar^3+\ldots\ldots+ar^{n-1}+ar^n \ldots\ldots\ldots (ii)$$

Subtracting equation (ii) from (i)

$$S_n-r\,S_n=a-ar^n$$

$$S_n(1-r)=a(1-r^n)$$

$$S_n=\frac{a(1-r^n)}{1-r} \text{ or } \frac{a(r^n-1)}{r-1} \ldots\ldots\ldots (5)$$

according as r is less than or greater than 1

With the help of the above two formulas, if any three of the five quantities a, r, n, L and S_n are known, the other two can be easily found.

If r is less than 1, the sum of the geometric series upto infinity will be given by

$$S\infty=\frac{a}{1-r} \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

since we know that any fraction less than 1 raised to the power infinity will become zero. In other words as n approaches infinity r^n will approach zero. Thus r^n will become zero of the formula

$$S_n=\frac{a(1-r^n)}{1-r}$$

converting it into the form (6) given above.

Example I :

Find the sum to 8 terms of the series 1, 2, 4, 8 . . . .

Solution :

Here $a=1$, $r=2$ and $n=8$. It is a G P.

$$S_n=\frac{a(r^n-1)}{r-1}$$

$$S_n = \frac{1\times(2^8-1)}{2-1}$$

$$= \frac{256-1}{1}$$

$$= 255$$

The 8th term will be $= ar^{n-1}$

$$= 1\times(2)^7$$

$$= 128$$

Example 2 :

Find the sum of the series $1, -\frac{1}{2}, \frac{1}{4}, -{}^1/_8, {}^1/_{16}, -{}^1/_{32}, \ldots\ldots$ to 10 terms.

Solution :

It is a G.P. where $a=1$, $r=-\frac{1}{2}$ and $n=10$.

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{1\times\{1-(-\frac{1}{2})^{10}\}}{1-(-\frac{1}{2})}$$

$$= \frac{1-(\frac{1}{2})^{10}}{1+\frac{1}{2}} = \frac{1-\frac{1}{1024}}{\frac{3}{2}}$$

$$= \frac{\frac{1023}{1024}}{\frac{3}{2}} = \frac{1023}{1024}\times\frac{2}{3}$$

$$= \frac{341}{512}$$

Example 3:

Find the sum of the series

$$1,\ \frac{3}{4},\ \frac{9}{16},\ \frac{27}{64},\ \frac{81}{256},\ \ldots\ldots\ldots\ldots \text{ to infinity}$$

Solution:

The given series is a G.P. as it can be written as:

$$1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\left(\frac{3}{4}\right)^4+\ldots\ldots\ldots \text{ to infinity}$$

Hence sum to infinity is given by

$$S_\infty = \frac{a}{1-r} \quad \text{where } a = 1, \text{ and } r = \frac{3}{4}$$

$$= \frac{1}{1-\frac{3}{4}}$$

$$= \frac{1}{\frac{1}{4}}$$

$$= 4$$

Note that this example can also be stated as:

Prove that the sum of the series

$1, \frac{3}{4}, \frac{9}{16}, \frac{27}{64}, \frac{8}{256}, \ldots\ldots\ldots\ldots$ to infinity

cannot exceed 4.

8.6. Geometric Mean

The *geometric mean* is the term between any two given terms of a geometric progression.

Thus if 'a' and 'b' are any two terms of a G.P. the geometric mean G will be such that :

$$\frac{G}{a} = \frac{b}{G} = \text{common ratio}$$

$$G^2 = ab$$

$$G = \sqrt{ab}$$

To insert a given number of geometric means between two terms, we may apply the formula. $L = ar^{n-1}$ but n here is the number of terms to be inserted plus 2.

Example 4 :

(i) Find the geometric mean between 3 and 192.

(ii) Insert 5 geometric means between 3 and 192.

Solution :

(i) $G = \sqrt{3 \times 192}$

$= \sqrt{576}$

$= 24$

(ii) Applying the formula $L = ar^{n-1}$, where $a = 3$, $L = 192$, and $n = 5+2 = 7$.

$$192 = 3r^{7-1}$$

$$r^6 = \frac{192}{3} = 64$$

$$= (2)^6$$

The required geometric means are :

$3 \times 2 = 6$

$6 \times 2 = 12$

$12 \times 2 = 24$

$24 \times 2 = 48$

$48 \times 2 = 96$

So the G.P. becomes 3, 6, 12, 24, 48, 96, 192.

It may be noted that the use of logarithm will be necessary when number of terms n of the G.P. is big as the value r^n cannot be determined easily. For example when $r=2$ and $n=100$, the value of 2^{100} can be easily determined by using logarithms which is otherwise very time consuming.

8.7. Application of a Geometric Progression

The use of logarithms and the amazingly increasing sum of a G.P. is illustrated in the following example.

Example 5 :

A company offers two alternatives for the payment of salary for the post of a high executive, Either one may receive Rs. 20,000 per months or 1 paisa on the first day of the month, 2 paisas on the second day, 4 paisas on the third day, 8 paisas on the fourth day and so on for the rest of the month i.e. each day getting the double of what he has received one day before. Which of the two alternatives should he prefer ?

Solution :

A layman would certainly prefer to accept a salary of Rs. 20,000 p.m. without any hesitation but a mathematician would not. On the basis of his calculation he would accept the second alternative as it accumulates to a huge sum of money. Guess, what the expected amount would approximately be. You will be surprised, rather amazed, to know the magnitude of the sum involved. Let us find it out mathematically.

Daily receipts expressed in paisas would form a geometric series 1, 2, 4, 8, 16, 32, 64 ..... upto 30 terms (supposing a 30 days month) whose first term $a=1$, $r=2$ and $n=30$.

Total receipts for the month will be the sum S of the series upto 30 terms given by :

$$S = 1+2+4+8+16+32+ \ldots\ldots \text{ upto 30 terms.}$$

$$= \frac{a(r^n - 1)}{r-1}$$

$$= \frac{1(2^{30}-1)}{2-1} = (2)^{30} - 1$$

Solution :

Substituting $S=675$, $C=500$ in the formula

$$S=C\,(1+r)$$
$$675=500\,(1+r)$$
$$=500+500r$$
$$500r=675-500$$
$$=175$$
$$r=\frac{175}{500}=0.35=35\%$$

Example 3 :

What is the cost of a steel cabinet which sells for Rs. 840 if the percentage mark-up on cost is 20%.

Solution :

Using the formula, we have

$$840=C\,(1+0.20)$$
$$=1.20\,C$$
$$C=\frac{840}{1.20}=\text{Rs }700$$

Let us study the situation when *mark-up is stated as a percentage on sales.* Let p be the percentage mark up on sales price. Then $p.S$ would denote the amount of profit or mark-up.

We know that

Profit = Sales Price — Cost Price.

$$p.S=S-C$$
$$C=S-p.S$$
$$C=S\,(1-p)$$

Example 4 :

After a mark-up of 30% on sales a watch sells for Rs. 225.

(i) What is its cost price ?

(ii) What is the percentage mark-up on sales if the cost price of the watch would have been Rs. 153 ?

Solution :

(i)
$$C=S\,(1-p)$$
$$C=225\,(1-0.30)$$
$$=225\times 0.70$$
$$=\text{Rs. }157.50$$

(ii)
$$C=S(1-p)$$
$$153=225\,(1-p)$$
$$=225-225p$$
$$225p=225-153=72$$
$$p=\frac{72}{225}=0.32=32\%$$

9.2. Simple Interest and Present Value

When a person borrows money he has to pay some amount as interest as a fee for the use of money. The money borrowed is called the *principal*. The sum of the principal and interest is known as the *amount*. The rate of interest is generally expressed as a percentage of the principal for a period of time which is usually one year. When interest is paid only on the sum borrowed it is called *simple Interest* but when the interest for each period is added to the principal for determining the interest of the subsequent period it is called *compound interest*. In this chapter we shall be dealing with simple interest only. If the sum of money borrowed called the principal or the present value is P, r the rate of interest expressed as a percentage per year, and t the time in years, then by the definition simple interest, denoted by I, should be equal to the principal multiplied by the rate multiplied by the time Thus.

$$I = Prt$$

This relationship can be used to determine any one of the four variables if the other three are known.

The total amount A to be paid at the end of t years is therefore given by :

$$\begin{aligned} A &= P + I \\ &= P + Prt \\ &= P(1 + rt) \end{aligned}$$

The principal or present value of an amount P at simple interest r can be written by expressing the above relation in terms of P in the following form :

$$P = \frac{A}{1 + rt} = A(1 + rt)^{-1}$$

Example 1 :

Mahmood borrowed Rs. 1.500 from Saleem for $3\frac{1}{2}$ years at a simple interest rate of 8% per annum How much Mahmood has to pay at the end of the period ?

Solution :

Here $P - 1{,}500$

$r = 08$

$t = 3\frac{1}{2} = 3.5$

Substituting these in the formula

$$\begin{aligned} A &= P(1 + rt) \\ &= 1{,}500\,[1 + (.08 \times 3.5)] \\ &= 1{,}500\,(1 + .28) \\ &= 1{,}500 \times 1.28 \\ &= \text{Rs. } 1{,}920 \end{aligned}$$

ıple 2 :

Find the present value of Rs. 3,500 due in four months at 6% interest per annum.

Solution :

$$P=\frac{A}{1+rt}$$

$$=\frac{3500}{1+(.06\times\frac{4}{12})}=\frac{3500}{1.02}$$

$$=\text{Rs. } 3{,}431.37$$

In computing interest it is customary to consider a 360-day year instead of a 365 or 366-day year. Thus 30 days will be considered as $^{30}/_{360}={}^{1}/_{12}$ of an ordinary year and so on The interest thus obtained is called ordinary interest but if it is based on 365 days it is called the exact interest.

Example 3 :

Find the ordinary interest on a loan of Rs. 2,500 at 10% simple interest from January 5 to February 10.

Solution :

January 5 to February 10, we have 36 days.

$$I=2{,}500\times(0.10)\times\frac{36}{360}$$

$$=\text{Rs. } 25$$

Generally the bank deducts interest if it extends short term loan not exceeding one year to its customers. The interest is deducted in advance and is calculated on the maturity value of the amount of the loan *i e.* on the amount which must be paid at the end of the period Thus, for a short term loan of Rs. 1,000 payable one year hence, the bank will pay Rs. 920 now, if the interest rate is 8%.

9.3. Simple Discount

Let A be the amount to be paid at maturity after time t at the simple interest rate of r percent per annum.

Then the simple interest I on maturity value A in time t is given by

$$I=Art$$

This must be subtracted from A to get the present value P.

$$P=A-Art$$

$$=A\,(1-rt)$$

9.4. Discounting Negotiable Instruments

Negotiable instrumens are written promises to pay money on a certain specific date. They are of two types, *viz.* non interest bearing and interest bearing. The basic principles of discounting a bill of exchange or short term note at a bank or at any other party are the same as those of obtaining a loan from a bank which deducts interest in advance. The following example will illustrate the procedure.

(a) Discounting non-Interest-Bearing Note

Example 4 :

After Khalid accepted a bill for Rs. 4,500, Hanif discounted it at National Bank Karachi on April 15. The maturity date of the bill was May 15. How much did Hanif receive if the bill was discounted at 8% ?

Solution :

The period of discount=30 days or $t = {}^{30}/_{360} = {}^{1}/_{12}$ year.

$A =$ Rs. 4,500 and $r = 8\%$

The discounted value

$$P = A\,(1 - rt)$$
$$= 4{,}500\left(1 - \frac{8}{100} \times \frac{1}{12}\right)$$
$$= 4{,}500\left(1 - \frac{1}{150}\right)$$
$$= 4{,}500 \times \frac{149}{150}$$
$$= \text{Rs. } 4{,}470$$

Example 5 :

A three-month non-interest-bearing note dated September 12, 1984 was discounted by Habib Bank Karachi on October 12, 1984 for Rs. 2,960 at 8%. Find the face value of the note.

Solution :

$P =$ Rs. 2,960, $r = 8\%$ and $t = {}^{60}/_{360} = {}^{1}/_{6}$ year

Face value $A = \dfrac{P}{1 - rt}$

$$= \frac{2{,}960}{\left(1 - \frac{8}{100} \times \frac{1}{6}\right)} = \frac{2{,}960}{\left(1 - \frac{1}{75}\right)}$$
$$= 2{,}960 \times \frac{75}{74}$$
$$= \text{Rs. } 3{,}000$$

(b) Discounting Interest-Bearing Note

We have seen in example 4 that the maturity value of the note was given on which the discount was computed. But in an interest-bearing note the maturity value of the note is not given. Thus in discounting an interest-bearing note we have to follow the two steps given below :

(i) Find out the maturity value from the face value of the note after adding the interest which would have been earned upto the maturity date at the given rate.

(ii) Find out the proceeds by discounting the maturity value obtained in step (i) at the discounting date.

Example 6 :

Mohsin had a note for Rs. 15,000 with an interest rate of 6%. The note was dated January 12, 1983 and the maturity date was 90 days after date. On January 27, 1983 he took the note to his bank which discounted it at a discount rate of 7%. How much did he receive ?

Solution :

Step-1 : Maturity value A is given by :

$$A = P(1+rt)$$
$$= 15000\left(1+\frac{6}{100}\times\frac{3}{12}\right)$$
$$= 15000\ (1+.015)$$
$$= 15000\times 1.015$$
$$= \text{Rs. } 15{,}225$$

Step-2 : Discounting this maturity value at 7% we have

$$P = A\ (1-rt)$$
$$= 15225\left(1-7\%\times\frac{75}{360}\right)$$
$$= 15225\left(1-\frac{7}{100}\times\frac{75}{360}\right)$$
$$= 15225\left(1-\frac{7}{480}\right)$$
$$= 15225\left(\frac{473}{480}\right)$$
$$= \text{Rs } 15{,}003$$

Example 7:

Suppose all other conditions are the same as in Example 6 except that Mohsin has advanced the loan for the same period to a party who agreed to pay back the loan by discounting it before the due date. What would Mohsin receive?

Solution:

Step 1 will be the same as in Example 6. Since the discounting is not done by any bank which deducts the interest on the final amount at the time of making the payment, we will have to find out the present value of Rs. 15,225 obtained in step 1.

It would therefore be given by:

$$P = A(1+rt)^{-1}$$
$$= 15{,}225\left(1+7\%\times\frac{75}{360}\right)^{-1}$$

Add Interest (3rd period) $+P\,(1+i)^2\,i$

Principal (3rd period end) $P\,(1+i)^2\,(1+i)=P\,(1+i)^3$

Continuing in this manner we will have the compound amount at the end of nth period $=P\,(1+i)^n$

Expressing the compound amount as 'S' we have

$$S=P\,(1+i)^n \qquad \ldots\ldots\ldots\ldots(1)$$

The factor $(1+i)^n$ is called the accumulating factor which is the compound amount in rupees when the principal is Re. 1

Example 1 :

Find out the compound amount and the compound interest at the end of 3 years on a sum of Rs. 20,000 borrowed at 6% compounded annually.

Solution :

We have to find S, when P= Rs. 20,000 ; $i=6\%=.06$ and $n=3$ periods (yearly)

$$S=P\,(1+i)^n$$
$$=20{,}000\,(1+.06)^3$$
$$=20{,}000\times(1.06)^3$$
$$=20{,}000\times 1.191016$$
$$=\text{Rs. } 23{,}820.32.$$

Compound Interest $= S-P$

$$=\text{Rs. } 23{,}820.32-\text{Rs. } 20{,}000$$
$$=\text{Rs. } 3{,}820.32$$

In this example we have $n=3$, and the value of $(1.06)^3$ can be calculated easily by simple multiplication. But it would be very difficult to find out the value of $(1+i)^n$ for higher values of n. In that case it would be advisable to use logarithms to determine the value of this factor.

Example 2 :

If Rs. 3,000 are invested at 6% interest compounded semi-annually what would it amount to at the end of 8 years ?

Solution :

To find S, when P=Rs. 3,000; $i=\frac{1}{2}$ of $6\%=.03$: $n=8\times 2=16$ periods

$$S=P\,(1+i)^n$$
$$=3{,}000\,(1+.03)^{16}$$
$$=3{,}000\,(1.03)^{16}$$

Using logarithms to find the value of $(1.03)^{16}$

Let $x=(1.03)^{16}$

$\log x = 16 \log (1.03),$

$= 16\times 0.012837$

$= 0.205392$

$x = \text{anti log } (0.205392)$

$= 1.60469$

Substituting the value, we get

S = Rs. 3,000 × 1.60469

= Rs. 4,814.07

10.3. Use of Compound Interest Tables

Compound interest and other related problems are so frequent in business organisations generally and financial institutions particularly that tabular aids have been constructed to make the computational work easier and at the same time more accurate. In each table periods are given horizontaly at the extreme left and different percentages per period on the top column. Table-3 relates to the compound amounts of Re 1 for n periods at i percent per period.

Using Table 3 for solving Example-2, the value of $(1+.03)^{16}$ is 1.6047064 at the crossing of n=16 row and i=3% column.

Therefore S = Rs. 3,000 × 1.6047064

= Rs. 4,814.12

It may be noted that the slight difference in the result is due to approximation.

10.4. Finding out the Number of Periods and the Interest Rate

The basic formula of compound interest can also be stated in terms of 'n' and 'i' with the help of which we can directly find out the number of periods or interest rate per period when other variables are known.

We know that,

$$S = P(1+i)^n$$

$$(1+i)^n = \frac{S}{P}$$

$n \log (1+i) = \log S - \log P$ (taking log of both sides)

Therefore $n = \dfrac{\log S - \log P}{\log (1+i)}$ ....(2)

This is the formula for number of periods when S,P and i are known

Also $n \log (1+i) = \log S - \log P$

$$\log (1+i) = \frac{1}{n} (\log S - \log P)$$

$$(1+i) = \text{Antilog}\left\{\frac{1}{n}(\log S - \log P)\right\}$$

Therefore $$i = \text{Antilog}\left\{\frac{1}{n}(\log S - \log P)\right\} - 1 \ldots\ldots(3)$$

This is the formula for rate of interest 'i' per period when S,P and n are known.

Example 3 :

In how many years a sum of Rs. 3,000 would amount to Rs. 4,814.07 at 6% interest compounded semi-annually ?

Solution :

$$P(1+i)^n = S$$
$$3{,}000\,(1+.03)^n = 4{,}814.07$$
$$(1+.03)^n = \frac{4{,}814.07}{3{,}000}$$
$$n \log(1.03) = \log 4{,}814.07 - \log 3{,}000$$
$$\text{Therefore } n = \frac{\log 4{,}814.07 - \log 3{,}000}{\log(1.03)}$$
$$= \frac{3.682513 - 3.477121}{0.012837}$$
$$= \frac{0.205392}{0.012837}$$
$$= 16 \text{ periods of six months.}$$
$$= 8 \text{ years.}$$

By the use of table :

$$\frac{S}{P} = \frac{4{,}814.07}{3{,}000} = 1.60469 \qquad \text{(amount for Re 1)}$$

In Table 3 read the figures under column 3% till you get a value approximate to 1.60469. It is against $n=16$, showing 16 periods as obtained above. Exact value may be found out by interpolation as we do in using logarithmic tables.

Example 4 :

At what rate of interest compounded semi-annually for 8 years will Rs. 3,000 amount to Rs. 4,814.07 ?

Solution :

$$P(1+i)^n = S$$
$$3{,}000\,(1+i)^{16} = 4{,}814.07$$
$$(1+i)^{16} = \frac{4{,}814.07}{3{,}000}$$
$$16 \log(1+i) = \log 4{,}814.07 - \log 3{,}000$$
$$= 0.205392$$

$$\log(1+i) = \frac{0.2053...}{16} = 0.01...$$

$$1+i = \text{antilog } (0.012837)$$
$$= 1.03$$

Therefore $i = 1.03 - 1$
$$= .03$$
$$= 3\% \text{ semi-annually}$$
$$= 6\% \text{ annual rate of interest}$$

As for Example 3, we can read out the required value directly from Table 3 by going along the row for $n=16$ till we get a value very approximate to 1.60469.

Example 5 :

Find the compound amount and the compound interest when Rs. 5,000 are invested for 3 years and 2 months at 6% compounded semi-annually.

Solution :

Here the invested period is more than 6 semi-annual periods. Therefore the solution will be worked out in two stages.

Stage (1) Find out the compound amount at the end of 3 years.

Find S, when P=Rs. 5,000; $i=.03$; $n=3\times2=6$ periods

$$S = 5{,}000\,(1+.03)^6$$
$$= 5{,}000\times1.194052$$
$$= \text{Rs. } 5{,}970.26$$

Stage (2) Find out the amount at simple interest if Rs. 5,970.26 are invested for 2 months at 6%

$$S = P\,(1+rt)$$
$$= 5{,}970.26\,(1+0.06\times\tfrac{2}{12})$$
$$= 5{,}970.26\,(1+0.01)$$
$$= 5{,}970.26\times1.01$$
$$= \text{Rs. } 6{,}029.96$$

The compound interest = Rs. 6.029.96—Rs. 5,000
= Rs. 1,029.96

10.5. Effective Rate of Interest

The *effective rate* of interest may be defined as the rate of interest compounded annually which gives the same amount of interest as obtained by a nominal rate compounded a number of times each year.

Example 6 :

Find out the effective rate of interest equivalent to the nominal rate of 8% compounded quarterly.

Solution :

Let x represent the effective rate of interest.

In one year Re. 1 will amount to Re $1+$Re $1. x$

At 8% compounded quarterly Re 1 will amount to $\left(1+\frac{.08}{4}\right)^4$

Therefore Re $1+$Re $1.x = \left(1+\frac{.08}{4}\right)^4$

$$
\begin{aligned}
x &= (1+.02)^4 - 1 \\
&= 1.082432 - 1 \\
&= 0.082432 \\
&= 8.24\%
\end{aligned}
$$

Thus the nominal rate of 8% compounded quarterly is equal to the effective rate of 8.24% per annum.

The formula for effective rate can therefore be stated as

$$e = (1+i)^m - 1$$

where e = the effective rate

i = interest per conversion period

and m = number of conversion periods in *one* year

10.6. Equivalent Rates

The rates of interest compounded over different conversion periods are said to be *equivalent* if the same pricipal invested at the respective rates accumulate the same compouud amount in a certain period of time.

Excmple 7 :

At what nominal rate compounded quarterly will a principal accumulate to the same amonnt as at 8% compounded semi-annually?

Solution :

Let the unknown nominal rate be x, then $i = \frac{x}{4}$ per quarter.

The effective rate will then be

$$e = \left(1+\frac{x}{4}\right)^4 - 1$$

In the second case $i = \frac{8\%}{2} = 4\%$ and the effective rate

$$e = (1+4\%)^2 - 1$$

Since the amount is the same, the two effective rates must be equal, Therefore

$$
\begin{aligned}
\left(1+\frac{x}{4}\right)^4 - 1 &= (1+4\%)^2 - 1 \\
\left(1+\frac{x}{4}\right)^4 &= (1+4\%)^2 \\
\left(1+\frac{x}{4}\right)^2 &= 1+4\% = 1.04
\end{aligned}
$$

$$1+\frac{x}{4} = \sqrt{1.04}$$

$$= 1.0198039$$

$$\frac{x}{4} = 0.0198039$$

$$x = 0.0792156$$

$$= 7.92\%$$

10.7. Depreciation by Reducing Balance Method.

The mathematical principles involved in the compound interest formula can by used to find out the constant rate of depreciation based on reducing balance method.

Let C be the original cost of the machinery and T be the trade-in or scrap value of the machinery after n years of useful life and r be the percentage rate of depreciation on the reduced balance each year.

Then,

Depreciation for 1st year $= Cr$

Residual value after 1st Year $= C-Cr=C(1-r)$

Depreciation for 2nd Year $= C(1-r)r$

Residual value after 2nd Year $= C(1-r)-C(1-r)r$

$= C(1-r)(1-r)$

$= C(1-r)^2$

Proceeding in this manner we get :

Residual value after 'n' years $= C(1-r)^n$

But since n years is the useful life of the machinery when its residual or trade-in value is T, therefore

$$C(1-r)^n = T$$

$$(1-r)^n = \frac{T}{C}$$

$$1-r = \sqrt[n]{\frac{T}{C}}$$

$$r = 1-\sqrt[n]{\frac{T}{C}}$$

Example 8 :

A plant of a company is purchased for Rs. 200,000 and it is estimated that at the end of the useful life of 10 years its scrap ualue will be only Rs. 25,000. What depreciation rate should be charged per year on reducing balances of the value of the asset ?

Solution :

$$r = 1-\sqrt[10]{\frac{25,000}{2,00,000}}$$

$$= 1-(0.125)^{\frac{1}{10}}$$

$$\text{Let } x = (.125)^{\frac{1}{10}}$$

$$\log x = \frac{1}{10}\log(.125)$$

$$= \frac{1}{10} \times \bar{1}.096910$$

$$= \frac{1}{10} \times (-10+9.096910)$$

$$= \bar{1}.909691$$

$$x = \text{antilog } \bar{1}.909691$$

$$= 0.81225$$

$$\text{Therefore } r = 1-0.81225$$

$$= 0.18775$$

$$= 18.78\%$$

PRACTICE SET 10-A

1. Mr. Mohsin has invested Rs. 25,000 at 6% compounded annualy. What amount would he receive after the expiry of 4 years?

2. Find the compound interest and compound amount of Rs. 8,000 at 8% compounded annually for 10 years.

3. A sum of Rs. 500 is deposited in a bank which pays interest at the rate of 6% compounded semi-annually. What will be the value of the deposit in 18 months?

4. Mr. Aslam borrowed Rs. 800 at 6% compounded semi-annully If he makes no intermediate payments, how much would he owe 8 years later?

5. On a saving bank account Bank-A pays 6% interest compounded annually, while Bank-B pays 6% interest compounded semi-annually. On a deposit of Rs. 1,000, how much more interest will be earned in 2 years at Bank-B as compared to Bank-A?

6. What would an investment of Rs. 3,000/- at 6% compounded quarterly would amount to in 5 years?

7. Find the compound amount and the compound interest on Rs. 2,000 invested for 3 years and 4 months at 6% compound semi-annually.

Chapter

11

Mathematics of Finance - III

ANNUITIES

11.1. Definitions

It may be observed that the compound interest problems basically deal with a single deposit or a loan advanced which remains invested for the entire period. There are few persons who have large sums available to invest in this way. It is however easier to attain their objective to have a good sum at the end of a period by making a series of regular deposits. This led to the introduction of annuities.

An *annuity* is series of payments (usually equal in amounts) which are made at regular intervals of time such as annually, semi-annually, quarterly or monthly. If the payments are made at the end of the payment periods the annuity is called an *ordinary annuity*. If the periodic payments of an annuity are made at the beginning of each interval the annuity is called *annuity due*.

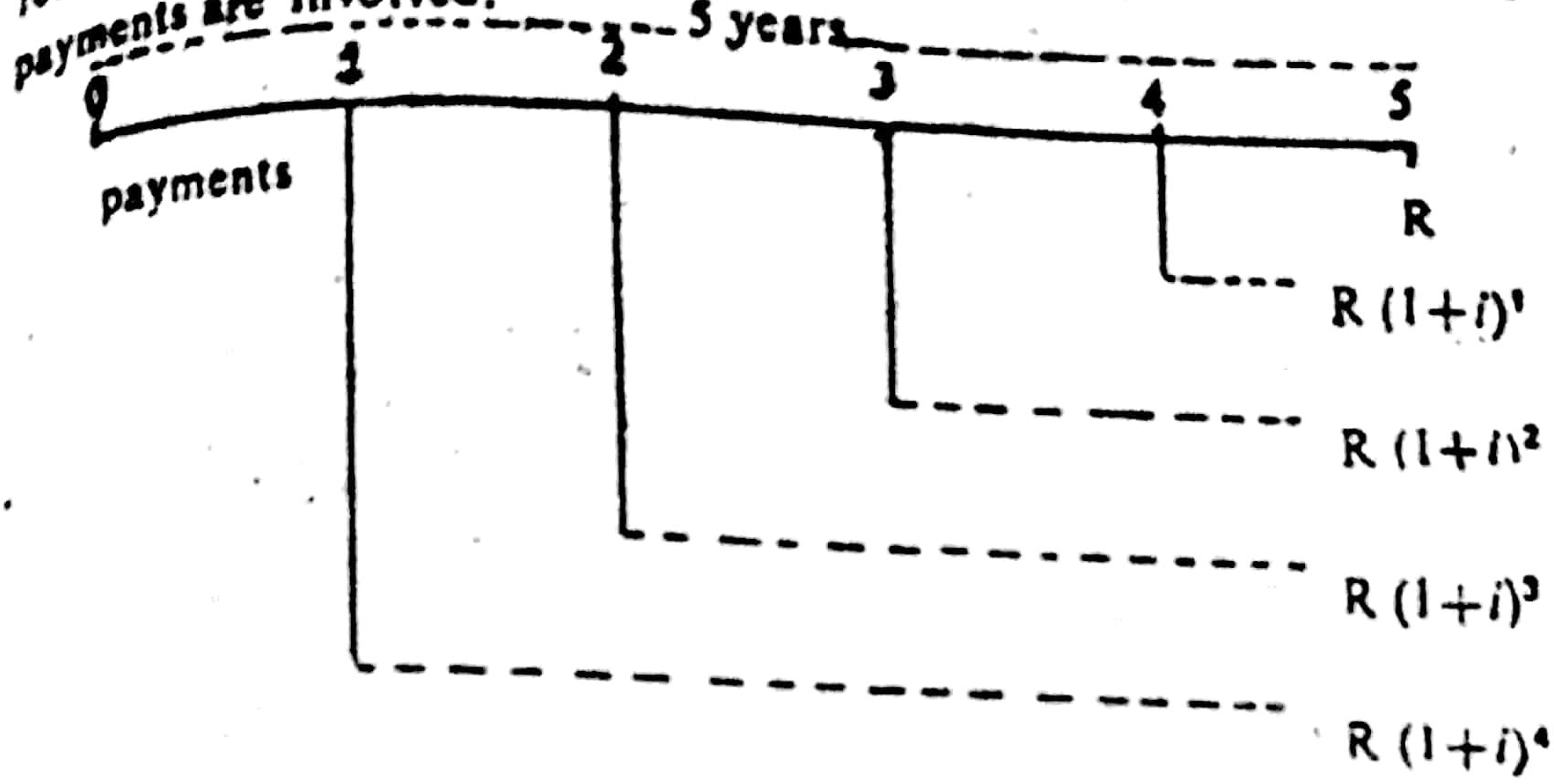
The time between two successive payment dates is called *payment period* and the time between the beginning of the first payment period and the end of the last payment period is called the *term* of the annuity.

SECTION A—SUM OR AMOUNT OF AN ORDINARY ANNUITY

11.2. Formula for Sum of Annuity

The amount of an annuity is the final maturity value of a series of equal payments at the end of the term of the annuity. The amount thus includes all periodic payments together with the compound interest thereon.

Let R be the payment per period, n the number of periods and i the interest rate per period of an annuity. We are interested in finding the (future) sum of this annuity which equals the sum of compound amounts of individual payments at the end of the period n. The following diagram will illustrate the principle in which the 5 yearly payments are involved.



Extending the period from 5 to 'n' years, each yearly payment 'R' will amount to the values given in the following arrangement :

1st Payment $= R(1+i)^4 = R(1+i)^{5-1} = R(1+i)^{n-1}$
2nd Payment $= R(1+i)^3 = R(1+i)^{5-2} = R(1+i)^{n-2}$
3rd Payment $= R(1+i)^2 = R(1+i)^{5-3} = R(1+i)^{n-3}$

. . . .

. . . .

$(n-2)$th Payment . . . . $= R(1+i)^2$
$(n-1)$th Payment . . . . $= R(1+i)^1$
nth Payment . . . . $= R$

It is obvious that since the first payment is made at the end of the first period it earns interest for $(n-1)$ periods and thus its compound amount is $R(1+i)^{n-1}$. Similarly the compound amount due to the second payment is $R(1+i)^{n-2}$ and continuing in this manner we can say that the compound amount due to $(n-1)$th payment is $R(1+i)$ and due to last i.e. the nth payment is $R(1+i)^0$ or simply R, because it earns no interest and mathematically any amount raised to the power zero equals one.

Summing up all the compound amounts we get the final sum 'S' of the annuity as

$$S=R+R(1+i)+R(1+i)^2+R(1+i)^3...+R(1+i)^{n-2}+R(1+i)^{n-1}$$

It is a geometric series of 'n' terms whose first term is R and the common ratio is $(1+i)$

Therefore by substituting these values in the original formula

$$S=\frac{a(r^n-1)}{r-1}$$

$$\text{We get } S=R\left[\frac{(1+i)^n-1}{i+1-1}\right]$$

$$S=R\left[\frac{(1+i)^n-1}{i}\right] \quad \text{.......................(1)}$$

This is the formula for obtaining the sum of an ordinary annuity just after the final payment.

11.3. Illustration of the Formula

The above formula can be illustrated by the following example :

Let us assume that we have made six deposits of Rs. 100 each in a saving account. The deposits on the first day of January for 6 consecutine years starting in 1975. Interest of 6% was added each time a deposit was made. What should be the total amount in the account immediately after the last deposit made on January 1, 1980 ?

Rs. 100/- deposited in 1980 with interest	=		Rs.	100.00
,, ,, deposited in 1979 with interest	become	$100 \times (1.06)$	,,	106.00
,, ,, ,, ,, 1978 ,, ,,	,,	$100 \times (1.06)^2$	,,	112.36
,, ,, ,, ,, 1977 ,, ,,	,,	$100 \times (1.06)^3$	,,	119.10
,, ,, ,, ,, 1976 ,, ,,	,,	$100 \times (1.06)^4$	,,	126.25
,, ,, ,, ,, 1975 ,, ,,	,,	$100 \times (1.06)^5$	,,	133.82
			Total Rs.	697.53

Since each partial amount belongs to a geometric progression of 6 term, with the first term 100 and a ratio of 1.06 the formula for the sum is $\dfrac{100\,[(1.06)^6-1]}{[1.06-1]} = \dfrac{100\,[(1.06)^6-1]}{0.06}$ This is exactly in accordance with the formula (1) of annuity. The reader is advised to calculate the sum by the formula and verify the result with the sum obtained above. Since direct computation as illustrated above is prohibitive when n is large, the best way is to apply the formula directly.

Let us see how this formula helps in solving the following different types of problems.

Example 1 :

Find the amount of an annuity of Rs. 500 payable at the end of each year for 10 years, if the interest rate is 6% compounded annually.

Solution :

We have R = 500, $n = 10$ and $i = .06$

$$S = R\left[\frac{(1+i)^n - 1}{i}\right]$$

$$= 500\left[\frac{(1+.06)^{10} - 1}{.06}\right]$$

$$= 500\left[\frac{(1.06)^{10} - 1}{.06}\right]$$

Find out the value of $(1.06)^{10}$ by logarithms.

Let $x = (1.06)^{10}$

$\log x = 10 \log (1.06)$

$= 10 \times .025306$

$= .25306$

$x = \text{Antilog } 0.25306$

$= 1.79085$

$\therefore S = 500 \left[\frac{(1.79085 - 1)}{.06} \right]$

$= 500 \times \frac{.79085}{.06}$

$= 500 \times 13.18083$

$= 6590.417$

$= \text{Rs. } 6,590.42$

11.4. Use of Annuity Table

Values of the sum of an ordinary annuity of Re. 1 are given in Table - 5. The symbol $s_{\overline{n}|i}$, which is read as "s angle n at i" is used for factor $\frac{(1+i)^n - 1}{i}$. It represents the sum of investments of Re. 1 per period for n periods at i percent per period.

$$S = R s_{\overline{n}|i} = R \left[\frac{(1+i)^n - 1}{i} \right]$$

Therefore by consulting Table - 5 against $i = 6\%$ and $n = 10$ our solution for the above example will be :

$$S = R s_{\overline{n}|i} = 500\ s_{\overline{10}|.06} = 500 \times 13.18079494$$

$= \text{Rs. } 6,590.40$

The difference is due to approximation

11.5. Finding R when S is known.

Example 2 :

A father at the time of birth of his daughter decides to deposit a certain amount at the end of each year in the form of an annuity. He wants that a sum of Rs. 20,000 should be made available for meeting the expenses of his daughter's marriage which he expects to be solemnised just after her 18th birthday. If the payments accumulate at 8% compounded annually, how much should he start depositing annually ?

Solution:

Here we have to find out R when

$S =$ Rs. 20,000, $n = 18$, and $i = .08$

In $S = R\left[\frac{(1+i)^n-1}{i}\right]$

$20000 = R\left[\frac{(1.08)^{18}-1}{0.08}\right]$

$R = \frac{20,000\times.08}{(1.08)^{18}-1} = \frac{1600}{(1.08)^{18}-1}$

Let $x = (1.08)^{18}$

$\log x = 18 \log (1.08)$

$= 18\times0.0334$

$= 0.6012$

$x =$ Anti-log (0.6012)

$= 3.99182$

Hence $R = \frac{1600}{3.99182-1} = \frac{1600}{2.99182}$

$=$ Rs. 534.79

The father should therefore deposit Rs. 534.79 each year for 18 years.

Also by Table - 5, we have

$\frac{20,000}{R} = s_{\overline{18}|.08} = 37.45024374$

$R = \frac{20,000}{37.45} =$ Rs. 534.05

Here again the difference is due to approximation.

11.6. Finding n when S is known.

Example 3 :

How many semi-annual payments of Rs. 100 each to an account in the form of an ordinary annuity will accumnlate Rs. 3,000 if the interest rate is 8% ?

Solution :

Here we have to find out n when

$S =$ Rs. 3,000.00

$i = \frac{0.08}{2} = 0.04$

$R =$ Rs. 100

In $S = R\left[\frac{(1+i)^n-1}{i}\right]$

$3,000 = 100\left[\frac{(1.04)^n-1}{0.04}\right]$

$(1.04)^n-1 = \frac{3,000\times0.04}{100}$

$(1.04)^n - 1 = 1.2$

$\therefore (1.04)^n = 2.2$

Taking log of both sides

$n \log(1.04) = \log 2.2$

$$\therefore n = \frac{\log 2.2}{\log(1.04)} = \frac{0.3424}{0.0170}$$

$= 20.1$ payments.

$= 10$ years (approximately)

The sum would accumulate almost to Rs. 3,000 in 20 six-monthly payments of Rs. 100 each.

Alternatively by the use of Table - 5 we have :

$$S = R s_{\overline{n}|i}$$

or $$s_{\overline{n}|i} = \frac{S}{R}$$

i.e. $$s_{\overline{n}|4} = \frac{3{,}000}{100} = 30$$

By consulting the table under 4% we find the value 29.778 very near to 30 against $n = 20$.

11.7. Finding 'i' when S is known.

It may be noted that if S, R and n are given the rate of interest per period i cannot be found with the help of logarithms as the value of the factor $[(1+i)^n - 1]$ cannot be determined if i is unknown. For such problems the only possible solution will be found by the use of annuity tables. Let us illustrate the method with the help of the following example :

Example 4 :

An annuity of Rs. 300 payable at the end of each quarter amounts to Rs. 13,200 in 8 years. What is the nominal rate of interest if it is compounded quarterly ?

Solution :

We have to find i when,

S = Rs. 13,200

R = Rs. 300-

$n = 8 \times 4 = 32$ periods

In the formula $S = R s_{\overline{n}|i}$, we have

$$13{,}200 = 300 \cdot s_{\overline{32}|i}$$

$$\therefore s_{\overline{32}|i} = \frac{13{,}200}{300} = 44$$

By following along the row $n = 32$ in Table 5 we do not find a value exactly equal to 44. So we have to interpolate by taking the nearest value less than 44 and the nearest value greater than 44. These are 37.4941 and 44.2270 against 1% and 2% columns.

2 %	44.2270	------	(i)
x %	44.0000	------	(ii)
1 %	37.4941	------	(iii)

$$x\% = 1\% + \left(\frac{44.0000 - 37.4941}{44.2270 - 37.4941}\right) \times (2\% - 1\%)$$

$$= 1\% + \frac{6.5059}{6.7329} \times 1$$

$$= 1\% + .969\%$$

$$= 1.969\%$$

Therefore the desired value of i is 1.969% per quarter or nominal interest rate is $1.969 \times 4 = 7.88\%$.

11 8. Sinking Fund for Repayment of Debt.

It sometimes happens that a debtor may like to depost equal amounts into a fund such that the deposits plus their interest is equal to the principal debt plus its interest. Such a fund is called a *sinking fund.* In such a situation, the interest earned by the debtor from the fund may or may not be equal to what he has to pay to his creditor.

Example 5 :

Mr. Sherwani wants to create a sinking fund to pay off his debt of Rs. 8,000 at 6% compounded annually in 5 years. If the sinking fund earns 8% interest compounded semi-annually, what should be his six-monthly deposit so as to enable him to pay off his debt at the end of 5 years ?

Solution :

The solution will be in two steps.

Step - 1.: To find the amount to be paid at the end of 5 years at 6% compounded annually.

$$A = 8{,}000\,(1+.06)^5$$

$$= 8{,}000 \times 1.33822558 \text{ (from Table-3)}$$

$$= \text{Rs. } 10{,}705.80$$

Step - 2 : The find R when S is Rs. 10,705.80 at 8% compounded semi-annually.

$$10{,}705.83 = R\,s_{\overline{10}|.04}$$

$$R = \frac{10{,}705.80}{s_{\overline{10}|.04}}$$

$$= \frac{10{,}705.80}{12.00610712} \qquad \text{(from Table-5)}$$

$$= \text{Rs. } 891.70$$

In some cases, the principal of a long term debt may be repaid on the maturity date, but the interest thereon is paid periodically when it is due. In such a situation, sinking fund will be established for the purpose of paying the principal of the debt at maturity and the periodic interest on the debt will not be paid out of the fund. The debtor will therefore be making two payments on the respective payment dates, one for the interest of the debt and the other for depositing in the fund.

Example 5a :

Supose in Example 5, the interest on the loan is to be paid at the end of each year while other facts remain unchanged.

Solution :

The annual payment of interest on the bebt

$= \text{Rs. } 8{,}000 \times 6\% = \text{Rs. } 480$

Six-monthly deposits in sinking fund will be given by

$$8{,}000 = R s_{\overline{10}|.04}$$

$$R = \frac{8{,}000}{s_{\overline{10}|.04}} = \frac{8{,}000}{12.00610712}$$

$$= \text{Rs. } 666.33$$

11.9. Depreciation by Sinking Fund Method

In this method, it is supposed that a sinking fund is maintained for the purpose of replacing a asset at the end of the useful life. The annual depreciation charges are exactly equal to the periodic increase in the sinking fund constituting the periodic deposit and the interest on the accumulated balance of the fund. Hence the depreciation charges are not equal each year but the accumulated depreciation charged is equal to the amount in the sinking fund (S_n) at the end of the useful life of asset.

If C is the total cost and T is the trade-in value of the asset, then

$$S_n = C - T$$

but $S_n = R s_{\overline{n}|i}$, therefore

$$R s_{\overline{n}|i} = C - T$$

$$R = \frac{C-T}{s_{\overline{n}|i}}$$

Example 6:

An asset purchased for Rs. 1,10,000 has a useful life of 5 years after which its trade-in value is expected to be Rs. 12,000. If the effective interest rate is 6% what will be the annual depreciation charges by the sinking fund method ?

Solution :

Here C= 1,10,000, T= 12,000, $i=6\%$ and $n=5$ years.

Accumulated Depreciation at the end of 5 years is given by

$$S_n = C-T = 1,10,000-12,000 = \text{Rs. } 98,000$$

Annual deposit in the sinking fund

$$R = \frac{98,000}{s_{\overline{5}|.06}}$$

$$= \frac{98,000}{5.63709296} \quad \text{(from Table 5)}$$

$$= \text{Rs. } 17,384.85$$

The annual depreciation charges are shown in columns of the following schedule :

DEPRECIATION SCHEDULE—SINKING FUND

(1) End of year	(2) Annual Deposit in Sinking Fund	(3) Interest Income from Fund (5) × 6%	(4) Increase in Fund = Annual Depreciation (2)+(3)	(5) Accumulated Sinking Fund = Accumulated Depreciation (From 4)	(6) Book Value of Asset Rs. 1,10,000—(5)
0	0	0	0	0	1,10,000 00
1	Rs. 17,384.85	0	17,384.85	17,384.85	92,615.15
2	17,384.85	1,043.09	18,427.94	35,812.79	74,187.21
3	17,384.85	2,148.78	19,533.63	55,346 42	54,653.58
4	17,384.85	3,320.79	20,705.64	76,052 06	38,947.94
5	17,384.82*	4,563.12	21,947.94	98,000 00	12,000.00
Total	86,924.22	11,075.78	98,000 00		

*Difference due to approximation corrected.

pays 6% compounded semi-annually what should be his semi-annual deposits into the sinking fund ?

12. Mr. Masood deposits Rs. 500 at the end of each quarter so as to accumulate a sum of Rs. 10,000 to purchase a refrigerator. If the interest rate is 5% compounded quarterly, how many such quarterly deposits he will have to make ?

13. Mrs. Jamil says that she will have Rs. 5,581.60 in her saving bank account at the end of 5 years. She deposits Rs. 80 one month from now and Rs. 80 thereafter at the end of each month. At what nominal interest rate compounded monthly has she calculated the interest ?

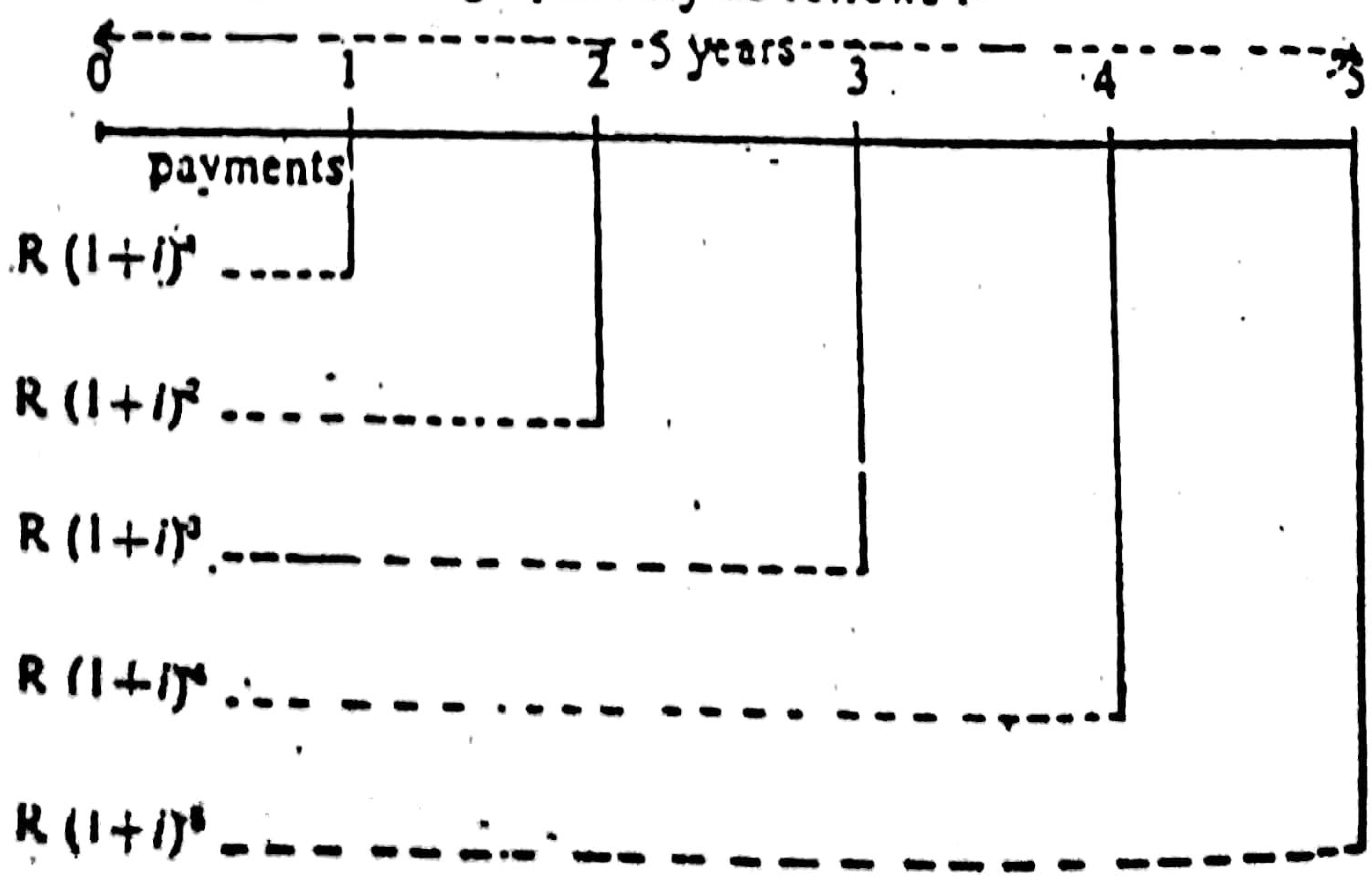
14. Zahid Enterprisers plan to exchange the old machine at the end of 4 years for a new one worth Rs. 2,50,000. The trade-in value of the old machine at that time is estimated to be Rs. 25,000. If the money can be invested at 6% compounded quarterly, how much must the company invest at the end of each quarter in order to make the exchange ?

SECTION B PRESENT VALUE OF AN ORDINARY ANNUITY

11.10. Formula for Present Value

We observed that in finding the amount of an annuity we started from a zero account and, by making payments into it, the account went on increasing so that the largest sum was at the end of the term. The idea behind the present value of an annuity is exactly the reverse. The amount is largest at the beginning and regular periodic equal payments are made till it is exhausted. This largest sum at the beginning of the term is called the *present value of an annuity*.

In deriving the formula for the present value of an ordinary annuity we will have to find out the present values of each periodic payment and add them up. Continuing the same technique of presentation, the problem can be represented graphically as follows :-



We have to sum up the present values of all the periodic payments given at the left end of the above diagram.

Let P be the sum of the present values of all periodic payments.

Then $P = R(1+i)^{-1} + R(1+i)^{-2} + --- + R(1+i)^{-n+1} + R(1+i)^{-n}$

It is a geometric series where first term is $R(1+i)^{-1}$, the common ratio is $(1+i)^{-1}$ and sum is P.

Therefore in $S_n = \dfrac{a(1-r^n)}{1-r}$ we have

$$P = \frac{R(1+i)^{-1}[1-(1+i)^{-n}]}{1-(1+i)^{-1}}$$

$$= \frac{\frac{R}{1+i}\left[1-(1+i)^{-n}\right]}{1-\frac{1}{1+i}}$$

$$= \frac{\frac{R}{1+i}\left[1-(1+i)^{-n}\right]}{\frac{1+i-1}{1+i}}$$

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Example 1 :

A loan of Rs. 94 is to be paid back in monthly instalments the first one starting after one month from the date of the loan. If the interest is charged at the rate of 24% per annum on the unpaid principal, what will be the amount of the monthly instalment ?

Solutoin :

Here R is required, when

$P =$ Rs. 94, $n = 12$ and $i = 2\% = .02$ per month,

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$94 = R\left[\frac{1-(1.02)^{-12}}{.02}\right]$$

Now by using logariths $(1.02)^{-12} = .7885$, we have

$$94 = R\frac{(1-.78849318)}{.02}$$

$$= \frac{R \times .2115}{.02}$$

$$R = \frac{94 \times .02}{.2115} = \text{Rs. } 8.89$$

Example 12 :

If Rs. 250 are deposited at the beginning of each quarter in a fund which earns interest at the rate of 8% compounded quarterly what will it amount to after the end of the year ?

Solution :

$$\text{Here } R = \text{Rs. } 250, \; i = \frac{8}{4} = 2\% \text{ per quater and } n=4$$

$$S_4 \text{(due)} = 250\,(s_{\overline{4+1}|.02} - 1) = 250\,(s_{\overline{5}|.02} - 1)$$

$$= 250\,(5.20404016 - 1) = 250 \times 4.20404016$$

$$= \text{Rs. } 1{,}051.01$$

Example 13 :

Mrs. Ahmad bought a sewing machine by paying Rs. 50 each month for 10 months, beginning from now. If money is worth 12% compounded monthly, what was the selling price of the machine on cash payment basis ?

Solution :

It is a problem on annuity due as the first payment starts now, the beginning of the first payment period.

$$\text{Here } R = \text{Rs. } 50, \; i = \frac{12\%}{12} = 1\%, \text{ and } n = 10$$

$$P \text{(due)} = 50\, a_{\overline{10-1}|.01} + 50$$

$$= 50\,(a_{\overline{9}|.01} + 1)$$

$$\doteq 50\,(8.56601758 + 1)$$

$$= 50 \times 9.56601758$$

$$= \text{Rs. } 478.20$$

(b) PERPETUITY

An annuity whose payments start on a certain date and continue indefintely is called a *perpetuity*. As the payments continue for ever, it is impossible to compute the amount of the perpetuity but its present value can be determined easily.

Using the same symbols as before, we start with the usual formula for the present value *viz* :

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$= \frac{R}{i}\,[1-(1+i)^{-n}] = \frac{R}{i}\left[1-\frac{1}{(1+i)^n}\right]$$

As $\frac{1}{1+i}$ is less than one, $\frac{1}{(1+i)^n}$ will tend to zero as n tends to infinity and thus :

$$P = \frac{R}{i}$$

Example 14 :

Pakistan Manufacturing Co. is expected to pay Rs. 4.80 every six months on a share of its stocks. What is the present value of a share if money is worth 8% compounded semi-annnally ?

Solution :

$$P = \frac{\text{Rs. } 4.80}{.04} = \text{Rs. } 120$$

Example 16 :

Find the present value of Karachi Toy Company share which is expected to earn Rs. 5.60 every six month, if money is worth 8% compounded quarterly.

Solution :

Since the time periods are different, we will have to calculate the equivalent rate of 8% compounded six-monthly first.

Rupee 1 at $x\%$ compounded six monthly will amount to $\left(1+\frac{x}{2}\right)^2$ and quarterly to $\left(1+\frac{.08}{4}\right)^4$ in one year.

$$\text{Therefore } \left(1+\frac{x}{2}\right)^2 = \left(1+\frac{.08}{4}\right)^4$$

$$1+\frac{x}{2} = (1+.02)^2$$

$$= 1.0404$$

$$\frac{x}{2} = .0404$$

$$x = .0808 = 8.08\%$$

$$\text{Thus present value } P = \frac{R}{i} = \frac{5.60}{0.0404}$$

$$= \text{Rs. } 138.61$$

PRACTICE SET 11-B

1. Find the present value of an annuity of Rs. 600 payable at the end of each year for 15 years if the interest rate is 5% compounded annually.

2. What is the present value of an annuity if the size of each payment is Rs. 200 payable at the end of each quarter for 8 years at an interest rate of 7% compounded quarterly ?

3. Calculate the present value of an annuity of Rs. 200 each month for 4 years at 6% compounded monthly ?

In the above example we have seen that the product of

$$[\,2,5,4\,] \quad \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = (2\times4) + (5\times2) + (4\times3)$$

and thus we can say that

$$[\,2,5,4\,] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2x + 5y + 4z$$

This suggests that an equation can be expressed in vector form and the equation $2x + 5y + 4z = 25$ can be written as

$$[\,2,5,4\,] \;.\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 25$$

Thus vectors provide a simple way of recording data and performing all the fundamental arithmetical operations on it.

16.3. Matrices

Matrix is an extension of the idea of vectors. It contains the basic concepts and characteristics of vectors. Suppose we have two vectors [2,5] and [3,7] arranged one below the other in the form $\begin{bmatrix} 2,5 \\ 3,7 \end{bmatrix}$. Such an arrangement of vectors is called a *matrix* and the components 2,5,3,7 are called *elements* of the matrix. Thus *a matrix may be defined as a rectangular array of numbers enclosed in brackets or in bold faced parentheses.* Generally matrices are represented by capital letters such as A, B and C shown below :

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}, \; B = \begin{bmatrix} 2,4 \\ 7\,6 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 1 \\ 3 & -3 \\ -5 & 4 \end{bmatrix}$$

A matrix is described by first stating its number of rows then its number of columns. This type of description is known as the *order* of a matrix. Thus in the above cases matrices A and B are 2 by 2 (written as 2×2) and that of matrics C is 3 by 2 (written as 3×2). In general if m represents the number of rows and n represents the number of columns, the order of the matrix would be $m \times n$ and we may call it an $m \times n$ matrix.

If $m = n$, the matrix is called a square matrix. Thus A and B are also called square matrix of order 2.

Solution :

$$C - D = \begin{bmatrix} 3 & 5 \\ 2 & 4 \\ 1 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 1 \\ 5 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-6 & 5-1 \\ 2-5 & 4-2 \\ 1-4 & 7-3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 4 \\ -3 & 2 \\ -3 & 4 \end{bmatrix}$$

(iii) **Multiplication**

(a) **Multiplication of a matrix by a real number (Scalar)**

When a matrix is multiplied by a real number, each element of the matrix is multiplied by that real number. The product so obtained is a matrix of the same order as that of the matrix multiplied.

Example 12 :

Multiply the 2×3 matrix $E = \begin{bmatrix} 2,5,7 \\ 3,1,4 \end{bmatrix}$ by 3

$$3 . E = 3 . \begin{bmatrix} 2 & 5 & 7 \\ 3 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times 2 & 3 \times 5 & 3 \times 7 \\ 3 \times 3 & 3 \times 1 & 3 \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 15 & 21 \\ 9 & 3 & 12 \end{bmatrix}$$

(b) **Multiplication of Matrix by another Matrix**

The product of two matrices is found by extension of the procedure of multiplying two vectors. It should be noted that *multiplication of two matrices will only be possible if the number of columns in the first matrix is equal to the number of rows in the second matrix.* If this condition is not satisfied multiplication will not be possible. Thus if the dimensions of the first matrix are $m \times n$ and those of the second matrix are $n \times p$ the multiplication will be possible and the result will be a matrix with dimensions $m \times p$.

1. To obtain any element in the product matrix, determine first the row and the column location of that element in the product matrix.

2. Multiply that row of the first matrix with that column of the second matrix. This will give the value of that element.

The principle is illustrated with the help of the following illustrations :

Example 13 :

1st Matrix		2nd Matrix		Product Matrix
$\begin{bmatrix} a \\ b \end{bmatrix}$	$\times$	$[c, d]$	=	$\begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}$
$\begin{bmatrix} 4 \\ 5 \end{bmatrix}$	$\times$	$[3, 2]$	=	$\begin{bmatrix} 12 & 8 \\ 15 & 10 \end{bmatrix}$

Example 14 :

1st Matrix		2nd Matrix		Product Matrix
$\begin{bmatrix} 1, & 3 \\ 2, & 4 \end{bmatrix}$	$\times$	$\begin{bmatrix} 4, & 5 \\ 6, & -2 \end{bmatrix}$	=	$\begin{bmatrix} 22, & -1 \\ 32, & 2 \end{bmatrix}$

WORKING :

1st Matrix	2nd Matrix	Calculations	Location in Product Matrix
Row 1 $[1, 3]$ $\times$	Col. 1 $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$	$(1\times4) + (3\times6) =$ 22	Row 1. Col. 1
Row 1 $[1, 3]$ $\times$	Col. 2 $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$	$(1\times5) + (3\times-2) = -1$	Row 1. Col. 2
Row 2 $[2, 4]$ $\times$	Col. 1 $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$	$(2\times4) + (4\times6) =$ 32	Row 2. Col. 1
Row 2 $[2, 4]$ $\times$	Col. 2 $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$	$(2\times5) + (4\times-2) =$ 2	Row 2. Col. 2

Example 15 :

1st Matrix		2nd Matrix		Product Matrix
$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 0 \\ 4 & 5 & 6 \end{bmatrix}$	$\times$	$\begin{bmatrix} -2 & 5 & 4 \\ 0 & 3 & -5 \\ -1 & 4 & 2 \end{bmatrix}$	=	$\begin{bmatrix} -4 & 22 & -7 \\ -6 & 21 & 2 \\ -14 & 59 & 3 \end{bmatrix}$

WORKING :

1st Matrix	2nd Matrix	Calculations	Location in Product Matrix.
Row 1 [1 3 2] ×	Col. 1 $\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$	$(1\times -2)+(3\times 0)+(2\times -1) = -4$	Row 1 Col. 1
Row 1 [1 3 2] ×	Col. 2 $\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$	$(1\times 5)+(3\times 3)+(2\times 4) = 22$	Row 1 Col. 2
Row 1 [1 3 2] ×	Col. 3 $\begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$	$(1\times 4)+(3\times -5)+(2\times 2) = -7$	Row 1 Col. 3
Row 2 [3 2 0] ×	Col. 1 $\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$	$(3\times -2)+(2\times 0)+(0\times -1) = -6$	Row 2 Col. 1
Row 2 [3 2 0] ×	Col. 2 $\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$	$(3\times 5)+(2\times 3)+(0\times 4) = 21$	Row 2 Col. 2
Row 2 [3 2 0] ×	Col. 3 $\begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$	$(3\times 4)+(2\times -5)+(0\times 2) = 2$	Row 2 Col. 3
Row 3 [4 5 6] ×	Col. 1 $\begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$	$(4\times -2)+(5\times 0)+(6\times -1) = -14$	Row 3 Col. 1
Row 3 [4 5 6] ×	Col 2 $\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$	$(4\times 5)+(5\times 3)+(6\times 4) = 59$	Row. 3 Col. 2
Row 3 [4 5 6] ×	Col. 3 $\begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$	$(4\times 4)+(5\times -5)+(6\times 2) = 3$	Row 3 Col. 3

16.5. Determinants

A determinant is a rectangular arrangement of numbers in rows and columns enclosed in two vertical lines. It is written in a manner similar to its associated form of square matrix except that the bracket of a matrix is replaced by two vertical lines. For example for an equivalent form of a 2×2 square matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, the determinant is denoted by $|A|$ and is written as

$$|A| = \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix}$$

This is a determinant of order 2. Similarly general form of a determinant of order 3 can be written as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

16.6. Valuation of a Determinant

The value of a determinant can be determined by two methods viz:

(i) By cross multiplication of the elements and,

(ii) By finding the minors.

(i) *Evaluation of Determinants by Cross Multiplication*

If a 2×2 square matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then the determinant of A is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad s_1 \swarrow \quad \searrow p_1$$

It has one primary diagonal and one secondary diagonal shown as p_1, and s_1 respectively.

Determinants of larger than 2×2 order have multiple primary diagonals and multiple secondary diagonals. For example in a determinant B of order 3×3 there are three primary diagonals p_1, p_2, and p_3 and three secondary diagonals s_1, s_2, s_3, as shown below :

$$|B| = \left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right.$$

$$s_1 \swarrow \quad s_2 \swarrow \quad s_3 \swarrow \qquad \searrow p_1 \quad \searrow p_2 \quad \searrow p_3$$

The numerical value of a determinant is found out by multiplying the elements lying on the primary diagonal and subtracting from them the product of the elements lying on the secondary diagonal.

Thus in a 2×2 determinant

$$|A| = p_1 - s_1 = a_{11}\,a_{22} - a_{12}\,a_{21}$$

and in a 3×3 determinant

$$|B| = (p_1 + p_2 + p_3) - (s_1 + s_2 + s_3)$$

$$= a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32}$$

$$- a_{13}\,a_{22}\,a_{31} - a_{11}\,a_{23}\,a_{32} - a_{12}\,c_{21}\,a_{33}$$

Example 16 :

(a) $|A| = \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix}$ (b) $|A| = \begin{vmatrix} 2 & -3 \\ -5 & 1 \end{vmatrix}$.

Solution :

(a) $|A| = \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix}$

s_1 p_1

$= (3\times 2) - (5\times 1)$

$= 6-5 = 1$

(b) $|A| = \begin{vmatrix} 2 & -3 \\ -5 & 1 \end{vmatrix}$

$= (2\times 1) + (-3) \times (-5)$

$= 2 - 15 = -13$

Example 17 :

Find the value of the determinant

$$|A| = \begin{vmatrix} 3 & -1 & 4 \\ 2 & 4 & -3 \\ 1 & 2 & 6 \end{vmatrix}$$

Solutiton :

$$|A| = \left|\begin{matrix} 3 & -1 & 4 \\ 2 & 4 & -3 \\ 1 & 2 & 6 \end{matrix}\right.\begin{matrix} 3 & -1 \\ 2 & 4 \\ 1 & 2 \end{matrix}$$

s_1 s_2 s_3 p_1 p_2 p_3

$$= p_1 + p_2 + p_3 - s_1 - s_2 - s_3$$
$$= (3\times4\times6) + (-1\times-3\times1) + (4\times2\times2) - (4\times4\times1)$$
$$- (3\times-3\times2) - (-1\times2\times6)$$
$$= 72 + 3 + 16 - 16 - (-18) + 12$$
$$= 72 + 3 + 16 - 16 + 18 + 12$$
$$= 105$$

It may be noted that in a 3×3 determinant neither the second nor the third primary and secondary diagonals pass through three elements. Therefore to facilitate calculation the first two columns of the determinants are repeated outside the vertical line of the determinant.

16.7. Concept of Minors or Cofactors

It may be noted that the cross multiplication method discussed above cannot be applied for determinants of order higher than 3. Evaluation of determinants of higher order can be done with the help of *minors*. *A minor is a determinant of order $(n-1)$ obtained from a determinant of order n.* Thus in a 3×3 determinant we may have 2×2 minors. A minor is specified by an element Thus the minor of the element a_{ij} located in the i^{th} row and j^{th} column in a determinant is denoted by A_{ij}. The minor A_{ij} is obtained by deleting both the i^{th} row and j^{th} column of the determinant. This can be seen from the following illustrations :

In a 2×2 determinant given by :

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \text{ has a minor } A_{11} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |a_{22}|$$

In a 3×3 determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ the minor is}$$

$$A_{21} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

A sign should be prefixed before a minor when it is used to evaluate a determinant. When $i+j$ is an even number the sign of the minor A_{ij} is positive but if the sum is odd the sign before the minor is negative $(-A_{ij})$.

Example 18 :

The minor of an element a_{11} is $+A_{11}$ as $1+1=2$

If $|A| = \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix}$, $a_{11} = 3$ and $+ A_{11} = \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix}$

$= + |5|$

The minor of element a_{21} is $- A_{21}$ as $2 + 1 = 3$ (odd)

If $|A| = \begin{vmatrix} 2 & 1 & 5 \\ 3 & 2 & 4 \\ 4 & 3 & 6 \end{vmatrix}$, $a_{21} = 3$, and therefore

$$A_{21} = - \begin{vmatrix} 2 & 1 & 5 \\ 3 & 2 & 4 \\ 4 & 3 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix}$$

A signed minor is also called a *cofactor* of the element A_{ij}.

(ii) *Evaluation of a Determinant by Minors (Co-factors)*

The following two steps are to be followed :

1. Multiply each element of any column (or any row) by its signed minor.

2. Add the n products obtained above. The sum will give the value of the determinant

Thus for a 2×2 determinant $|A|$ is given by

$|A| = a_{11} A_{11} - a_{21} A_{21}$

For example,

$$|A| = \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} = 3 \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} - 1 \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix}$$

$= 3|5| - 1|2| = 15 - 2 = 13$

Example 19 :

Evaluate the following determinant by minors :

$$|A| = \begin{vmatrix} 3 & 1 & 4 \\ 1 & 2 & 3 \\ 6 & 4 & 5 \end{vmatrix}.$$

Solution :

$$|A| = \begin{vmatrix} 3 & 1 & 4 \\ 1 & 2 & 3 \\ 6 & 4 & 5 \end{vmatrix} = 3 \begin{vmatrix} 3 & 1 & 4 \\ 1 & 2 & 3 \\ 6 & 4 & 5 \end{vmatrix}$$

$$-1 \begin{vmatrix} 3 & 1 & 4 \\ 1 & 2 & 3 \\ 6 & 4 & 5 \end{vmatrix} + 6 \begin{vmatrix} 3 & 1 & 4 \\ 1 & 2 & 3 \\ 6 & 4 & 5 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & 4 \\ 4 & 5 \end{vmatrix} + 6 \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$

$= 3\,(10-12) - 1\,(5-16) + 6\,(3-8)$
$= 3 \times (-2) - 1 \times (-11) + (6 \times -5)$
$= -6 + 11 - 30$
$= -25$

Determinants of higher order may be obtained by repeating the process explained above each time decreasing the order of the determinant by one.

16.8 Inverse of a Matrix

We know that the reciprocal or inverse of a number a is $1/a$ or a^{-1}. This holds goods for all the numbers. Similarly the inverse relationship exists between a matrix and the inverse of the matrix in all square matrices.

If A is a square matrix, then A^{-1} will be its inverse such that

$$A^{-1} \cdot A = A \cdot A^{-1} = I$$

where I is the *identity matrix* of the same order as A and A^{-1}.

An identity matrix is a square matrix having all elements of its principal diagonal (the line from the upper left corner of the matrix to the lower corner) equal to 1 and the rest of the element zero.

For example the 2×2 indentity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the 3×3 identity matrix,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In matrix algebra an identity matrix behavers in the same manner as 1 in ordinary algebra when operation of multiplication is performed Thus A. I. = I.A = A is true for all matrices A. This can be seen from the following example.

Example 20 :

Let a 2x2 matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$ then

$$A.I = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (3\times1) + (2\times0) & (3\times0) + (2\times1) \\ (1\times1) + (5\times0) & (1\times0) + (5\times1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} = A$$

16.9. Methods of Finding Inverse of a Square Matrix

There are two methods which are commonly used to find out the inverse of a square matrix *viz*.

(i) The Basic Method

(ii) By the use of Determinants.

(i) The Basic Method

The inverse of a square matrix (if it exists) can be found by placing the corresponding identity matrix adjacent to original matrix A. The required row operations are performed on this combined new matrix until the matrix which was A is converted into an identity matrix and the new matrix formed at the position of identity matrix becomes the inverse of the matrix A. This procedure is explained with the help of the following example.

Example 21 :

Find the inverse of $\begin{bmatrix} 4 & -5 \\ 3 & -2 \end{bmatrix}$

Solution :

Place the 2×2 identity matrix next to the original matrix to form the 2×4 matrix

$$\begin{bmatrix} 4 & -5 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{bmatrix}.$$

To reduce the original matrix to the identity matrix multiply the first row by 1/4 to obtain

$$\begin{bmatrix} 1 & -5/4 & 1/4 & 0 \\ 3 & -2 & 0 & 1 \end{bmatrix}$$

Multiply the first row by -3 and add the result to the second row to get

$$\begin{bmatrix} 1 & -5/4 & -1/4 & 0 \\ 0 & 7/4 & -3/4 & 1 \end{bmatrix}$$

Multiply the second row by 4/7 to get

$$\begin{bmatrix} 1 & -5/4 & 1/4 & 0 \\ 0 & 1 & -3/7 & 4/7 \end{bmatrix}$$

Multiply the second row by 5/4 and add the result to the first row to obtain

$$\begin{bmatrix} 1 & 0 & -2/7 & 5/7 \\ 0 & 1 & -3/7 & 4/7 \end{bmatrix}$$

Thus the inverse of

$$\begin{bmatrix} 4 & -5 \\ 3 & -2 \end{bmatrix} \text{ is } \begin{bmatrix} -2/7 & 5/7 \\ -3/7 & 4/7 \end{bmatrix}$$

(ii) **By Determinants**

The method of finding the inverse of a matrix can be simplified by using the determinants. Let the general form of a 2×2 matrix A be written in the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

If $A^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ exists, we will have $A A^{-1} = I$ in the form of

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplying the two matrices on the left side and equating corresponding elements four equations will be obtained. By solving the four equations, the following relationships can be obtained

$$b_{11} = \frac{a_{22}}{|A|} \qquad b_{12} = \frac{-a_{12}}{|A|}$$

$$b_{21} = \frac{-a_{21}}{|A|} \qquad b_{22} = \frac{a_{11}}{|A|}$$

and thus we have

$$A^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\begin{bmatrix} \frac{a_{22}}{|A|} & \frac{-a_{12}}{|A|} \\ \frac{-a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\text{where } |A| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \; a_{22} - a_{12} \, . \, a_{21} \neq 0$$

The inverse of a 2×2 matrix may also be written in terms of signed minors as follows:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & -A_{21} \\ -A_{12} & A_{22} \end{bmatrix}$$

since
$$A_{11} = a_{22} \quad ; \quad -A_{21} = -a_{12}$$
$$-A_{12} = -a_{21} \quad ; \quad A_{22} = a_{11}$$

This shows that if $|A| \neq 0$, then the inverse matrix A^{-1} exists. Similarly the inverse of a 3×3 square matrix can be written in terms of signed minors as

$$A^{-1} = \begin{bmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & -A_{32} \\ A_{13} & -A_{23} & A_{33} \end{bmatrix}$$

Example 22 :

Find the inverse of matrix $[A] = \begin{bmatrix} 4 & -5 \\ 3 & -2 \end{bmatrix}$ by signed minors.

Solution :

$$|A| = \begin{vmatrix} 4 & -5 \\ 3 & -2 \end{vmatrix} = -8 + 15 = -7$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & -A_{21} \\ -A_{12} & A_{22} \end{bmatrix}$$

$$= 1/7 \begin{bmatrix} -2 & 5 \\ -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2/7 & 5/7 \\ -3/7 & 4/7 \end{bmatrix}$$ as obtained before in Example 21

Example 23 :

Find the inverse of matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ by signed minors.

Solution :

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} \begin{matrix} 1 & -2 \\ 1 & 1 \\ 2 & 3 \end{matrix}$$

$$= (1\times1\times1) + (-2\times1\times2) + (1\times1\times3) - (1\times1\times2)$$
$$- (1\times1\times3) - (-2\times1\times1)$$
$$= 1 - 4 + 3 - 2 - 3 + 2$$
$$= -3$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & -A_{32} \\ A_{13} & -A_{23} & A_{33} \end{bmatrix}$$

$$= -1/3 \begin{bmatrix} \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} \end{bmatrix}$$

$$= -1/3 \begin{bmatrix} -2 & 5 & -3 \\ 1 & -1 & 0 \\ 1 & -7 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -5/3 & 1 \\ -1/3 & 1/3 & 0 \\ -1/3 & 7/3 & -1 \end{bmatrix}$$

16.10. Solution of Linear Equations

The usual elimination method we have studied in Chapter 5. Here we shall deal with the following three special methods of solving linear equations :—

(i) By Matrix

(ii) By Inverse of Matrix

(iii) By Determinants

(i) By Matrix

Let us consider the following set of equations :

Example 24 :

Solve :

$$4x + 5y = .$$

$$3x + 2y = 12$$

Solution.

In the matrix form the equations can be expressed as

$$\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 23 \\ 12 \end{bmatrix} \quad \ldots\ldots \quad \text{(i)}$$

The three matrices in equation (i) are called coefficients, variables and constants respectively. For the sake of convenience we may write the variables at the top of their respective coefficients as follows :

$$\begin{array}{cc} x & y \\ \end{array}$$
$$\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 23 \\ 12 \end{bmatrix} \quad \ldots\ldots\ldots \quad \text{(ii)}$$

PRACTICE SET 5-A

1. Solve for x the following equations :

(i) $3x+5=11$ (ii) $\frac{x}{2}-1=3$

(iii) $4x+(-7)=2x-(-1)$ (iv) $\frac{x}{5}-\frac{1}{3}=\frac{x}{3}+\frac{1}{5}$

(v) $(x+2)^2=(x-5)^2+7$ (vi) $\frac{x}{2}-\frac{1}{4}=\frac{3x}{4}+1$

2. What number increased by 25 gives 83 ?

3. The Weaving Department of a factory occupies 3.5 times as much floor space as the Shipping Department. If the Weaving Department occupies 4,200 square feet, what is the space occupied by the Shipping Department ?

4. Heat and Electricity together cost a company Rs. 1.080/- for the month of January. If the consumption for heating purposes is three times as much as light, how much each expense cost to the company ?

5. A group bonus is to be divided among four employees in the ratio of their basic salaries which are in the ratio of 7, 4, 2 and 5 respectively. If a total bonus of Rs. 540/- is to be divided among them what would each employee receive ?

6. A typewriter is sold for 1,500/-. The gross profit is 2/3 of the cost, what are the cost and the gross profit ?

SECTION B—SIMULTANEOUS EQUATIONS

5.4. Form

We have discussed simple equations with only one unknown. Simultaneous equations are a set of two or more equations with two or more variables. This section will exclusively deal with such equations.

Let us start with an equation of two variables x and y.

$$2x+3y=24$$

It means that the sum of 2 times x and three times y is equal to 24. To make it realistic in a statement form we can say that : how many kilos of tomatoes and peas can be purchased with Rs. 24 in hand if the prices of tomatoes and peas are Rs. 2 and Rs. 3 per kilo respectively ?

Arbitrarily we may assign different values to x and y to satisfy the equation. A set of values is given below :

$x=0$	3	12	9	-3	15
$y=8$	6	0	2	10	-2

If these points are plotted on a graph it may be seen that all such points will be on one and the same straight line. For this reason such equations are called *linear* equations. As the power of each variable is *one*, it is also called an equation of the *first degree*. If the power of any variable is *two* the equation is said to be a *second degree equation* and so on. The graph of such equations will not be a straight line but will result in a curve. (Graphs are discussed in the next chapter).

We have seen that the equation $2x+3y=24$ can be satisfied by an unlimited pair of values all lying on a straight line. But, in *general*, if there are two linear equations in two unknowns, there is only one solution for each unknown which satisfies both the equations. The two equations are called *independent equations*. If the two equations can be reduced to the same form, they are said to be *dependent*. Thus $x+y=4$ and $2x+2y=8$ are dependent equations because the latter can be reduced to the former by dividing it by 2. If two equations are such that no common solution exists, they are said to be *inconsistent* equations. Thus $x+y=3$ and $x+y=8$ are inconsistent having no point in common. Geometrically they are parallel to each other.

5.5. Solution of Simultaneous Equations

Let us start with a set of two linear equations and build up the necessary steps to arrive at the required solution.

Example 1

Solve

$$2x+5y=30 \quad \ldots \quad (1)$$
$$3x-2y=7 \quad \ldots \quad (2)$$

Solution :

The fitst step is to eliminate one of the two variables x or y and get the simple equation in terms of the other variable. This will enable us to get the value of this variable.

The second step would be to substitute this value in any of the original equations and get the value of the second variable.

The entire procedure is explained as follows :

1. Eliminate by subtraction or addition.

(a) *By subtraction* (elimination of x)

Multiply equation (1) by 3 and (2) by 2

$$6x+15y=90 \quad \ldots \quad (3)$$
$$6x-4y=14 \quad \ldots \quad (4)$$

Subtract (4) from (3)

$$19y=76$$
$$y=\frac{76}{19}=4$$

(b) *By addition* (elimination of y)

Multiply equation (1) by 2 and (2) by 5

$$4x+10y=60 \quad \ldots \quad (3)$$
$$15x-10y=35 \quad \ldots \quad (4)$$

Add (3) and (4)

$$19x=95$$

$$x=\frac{95}{19}=5$$

2. By substitution of the value of one variable obtained in step 1 in any one of the original equations.

By (a) above $y=4$. Put this value in equation (1)

$$2x+(5\times 4)=30$$

$$2x+20=30$$

$$2x=10$$

$$x=5$$

If (b) is adopted $x=5$ should be substituted in (1) to get the value of y.

The solution set is therefore $x=5$, $y=4$.

(Note that if the eliminating variable has positive sign in both equations the elimination will be affected by subtraction but if the signs are opposite it will be done by addition).

5.6. Equations with Three Unknowns

So far we have noticed that for solving a simple equation of one variable only one equation is required and for solving completely a set of equations of two variables, two equations are necessary. By inductive reasoning, we can draw the conclusion that for solving three variables completely a set of three equations is required, for four variables four equations and so on. In general, 'n' variable would require a set of 'n' equations to solve them completely if at all any common solution exists.

The procedure of solution will be the same. In each step one variable is eliminated till we are left over with only one varivble. As the number of variables increase the procedure becomes more and more cumbersome.

Example 2 :

Find the solution set of the system of equations :

$$2x+3y+z=13 \quad \ldots \quad (1)$$

$$3x+2y+4z=17 \quad \ldots \quad (2)$$

$$4x+5y+2z=24 \quad \ldots \quad (3)$$

Solution :

Elimination of x from equations (1) and (2)

$$6x+9y+3z=39 \quad \ldots \quad (4)$$

$$6x+4y+8z=34 \quad \ldots \quad (5)$$

Subtract (5) from (4)

$$5y-5z=5 \quad \ldots \quad (6)$$

Similarly from equations (2) and (3)

$$12x+8y+16z=68 \quad \ldots \quad (7)$$

$$12x+15y+6z=72 \quad \ldots \quad (8)$$

Subtract (7) from (8)

$$7y-10z=4 \quad \ldots \quad (9)$$

Elimination of y from equations (6) and (9)

$$35y-35z=35 \quad \ldots \quad (10)$$

$$35y-50z=20 \quad \ldots \quad (11)$$

Subtract (11) from (10)

$$15z=15$$

$$z=1$$

Substituting the value of z in equation (6)

$$5y-(5\times1)=5$$

$$5y=10$$

$$y=2$$

Substituting the value of $y=2$ and $z=1$ in equation (1)

$$2x+(3\times2)+1=13$$

$$2x+7=13$$

$$2x=6$$

$$x=3$$

The solution set is therefore $x=3$, $y=2$ and $z=1$.

5.7. Two equations in Three Unknowns

We have seen that three equations are required to determine the values of three variables completely. Let us now see how far we succeed in giving arbitrary values to one variable and getting corresponding values of other two variables. Thus instead of getting one set of solution for all variables, we will get several such sets. In solving some of the business problems such situations do often arise and the businessman selects that set of solution which optimizes his objective. This method is used in solving problems by linear programming technique discussed in Chapter 17.

Example 3

Find the solution set of the system :

$$2x+3y+z=4 \quad \ldots \quad (1)$$

$$x-y+2z=5 \quad \ldots \quad (2)$$

Solution :

Multiply equation (2) by 2

$$2x-2y+4z=10 \quad \ldots \quad (3)$$

Subtract (1) from (3)

$$-5y+3z=6$$

$$3z=5y+6$$

$$z=\frac{5y}{3}+2 \quad \ldots \quad (4)$$

Substituting the value of z obtained in equation (2)

$$x-y+2\left(\frac{5y}{3}+2\right)=5$$

$$x-y+\frac{10y}{3}+4=5$$

$$x+\frac{7y}{3}=1$$

$$x=\frac{-7y}{3}+1 \quad \ldots \quad (5)$$

The values of x and z expressed in terms of y and the constant obtained in equation (4) and (5) satisfy both the equations no matter what value is given to y. Let $y=c$ where c is any arbitrary real number. Then equations (4) and (5) will become

$$x=\frac{-7c}{3}+1$$

and $$z=\frac{5c}{3}+2$$

The solution set in terms of c will be :

$$x=\frac{-7c}{3}+1$$

$$y=c$$

$$z=\frac{5c}{3}+2$$

Different values of this solution set can be obtained by giving arbitrary values to c. For example, if $c=0$, then $x=1$, $y=0$ and $z=2$. Thus (1, 0, 2) is a member of the solution set.

If $c=6$, x and z will be -13 and 12 respectively. Thus $(-13, 6, 12)$ is yet another member of the solution set. In this manner innumerable such solution sets can be obtained.

5.8. Statement Problems in Two Unknowns :

Example 4 :

The cost of 3 dozen eggs and four dozen oranges is Rs. 27 and the cost of 4 dozen eggs and 5 dozen oranges is Rs. 35. What is the cost per dozen of each item ?

Solution :

Let x be the cost in Rupees of one dozen eggs and y be the cost in Rupees of one dozen oranges. Then

$$3x+4y=27 \quad \ldots \quad (1)$$

$$4x+5y=35 \quad \ldots \quad (2)$$

For eliminating x multiply (1) by 4 and (2) by 3

$$12x+16y=108 \quad \ldots \quad (3)$$

$$12x+15y=105 \quad \ldots \quad (4)$$